BULLETIN

OF THE

AMERICAN MATHEMATICAL SOCIETY

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VOLUME XXXVII, NUMBER 6

JUNE, 1931

PUBLISHED BY THE SOCIETY

MENASHA, WIS., AND NEW YORK

1931

PUBLISHED MONTHLY

Whole No. 381 $9.00 a Year

Entered as second class matter at the post office at Menasha, Wis., under the act of August 24, 1912.

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LINEAR q-DIFFERENCE EQUATIONS*

BY C. R. ADAMS

1. *Introduction.* Recent years have seen the publication by Nörlund,† Carmichael,‡ and Pincherle§ of several papers whose aim is to describe the state of development attained by the difference calculus and the theory of difference equations. Scant attention, however, has been paid to the subject of q-difference equations, which not only is closely kin to, but may properly be regarded as a part of, the field of difference equations. It may therefore be of some interest to present here a brief summary of what has been done in the theory of linear q-difference equations, together with indications of certain extensions.

The linear difference equation

$$\sum_{i=0}^{n} a_i(x) f(x + n - i) = b(x) \tag{1}$$

is a particular case of a functional equation which is linear in the unknown function f, and in which the argument of f is repeatedly subjected to the non-singular linear fractional substitution $\{x, (ax+b)/(cx+d)\}$. This more general type of functional equation, however, can always be reduced by means of a linear fractional transformation of the independent variable x to one of two normal forms, the first being (1) and the second

$$\sum_{i=0}^{n} a_i(x) f(q^{n-i}x) = b(x). \tag{2}$$

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* An address presented to the Society at the request of the program committee, February 28, 1931.

† Nörlund, *Sur l'état actuel de la théorie des équations aux différences finies*, Bulletin des Sciences Mathématiques, (2), vol. 44 (1920), pp. 174–192, 200–220; *Neuere Untersuchungen über Differenzengleichungen*, Encyklopädie der Mathematischen Wissenschaften, vol. II C 7 (1923), pp. 675–721. See also *Vorlesungen über Differenzenrechnung*, 1924, 9+551 pp.

‡ Carmichael, *The present state of the difference calculus and the prospect for the future*, American Mathematical Monthly, vol. 31 (1924), pp. 169–183.

§ Pincherle, *Il calcolo delle differenze finite*, Bollettino dell' Unione Matematica Italiana, vol. 5 (1926), pp. 233–242, and Atti della Società per il Progresso delle Scienze, vol. 15 (1927), pp. 153–162.

Essentially this fact was proved by Pincherle* in 1880, although this precise formulation of the statement appears to have been first made by Stridsberg† in 1910. The normal form is of the type (1) or of the type (2) according as the double points of the substitution $x'=(ax+b)/(cx+d)$ are coincident or distinct. Hence it may be said that in general the functional equation of more general type is reducible by a linear fractional transformation to the normal form (2), while in particular it may be so reducible to the normal form (1).

The equation (2) is commonly termed a linear q-difference equation,‡ and its close relationship to the equation (1) is manifest. Indeed it is apparent that by the transformation $x=q^z$ the equation (2) is carried over into an equation of type (1); consequently if solutions can be found for the equation (2) when the coefficient functions $a_i(x)$ and $b(x)$ are of a certain class of functions of x, the problem of obtaining solutions of (1) when the coefficients belong to the same class of functions of the argument q^z is solved. Likewise the inverse transformation carries the equation (1) into one of type (2).

The equation (2) is said to be of *order* n, homogeneous or non-homogeneous according as $b(x)$ is or is not identically zero; *for brevity we shall denote the respective cases by* $(2h)$ *and* $(2n)$. It is clear that the substitution $q'=1/q$ immediately reduces either of the cases $|q|>1$, $|q|<1$ to the other; the theory of the equation for $|q|=1$ differs essentially from that for $|q|\neq 1$ and unless otherwise specified it is to be understood throughout this paper that $|q|$ is $\neq 1$. The cases in which the coefficients $a_i(x)$ and $b(x)$ are of a certain character at $x=0$ and of the same character at $x=\infty$ are reducible each to the other by the substitution $x'=1/x$; we shall therefore in general restrict ourselves to the

* Pincherle, *Ricerche sopra una classe importante di funzioni monodrome*, Giornale di Matematiche, vol. 18 (1880), pp. 92–136.

† Stridsberg, *Contributions à l'étude des fonctions algébrico-transcendentes qui satisfont à certaines équations fonctionnelles algébriques*, Arkiv för Matematik, Astronomi och Fysik, vol. 6, no. 15, 1910, 31 pp.; vol. 6, no. 18, 1910, 25 pp. See also Carmichael, *The general theory of linear q-difference equations*, American Journal of Mathematics, vol. 34 (1912), pp. 147–168.

‡ The name *geometric difference equation* has been suggested; see Ryde, *A contribution to the theory of linear homogeneous geometric difference equations (q-difference equations)*, (Dissertation, Lund), 1921, 45 pp.

former. The variable x is in general understood to be complex, and the equation

$$(3) \qquad a_0(0)\rho^n + a_1(0)\rho^{n-1} + \cdots + a_{n-1}(0)\rho + a_n(0) = 0,$$

is commonly called the *characteristic equation* of $(2h)$ for $x=0$.

It should be observed that if we define

$$\Delta^0 f(x) = f(x),\ \Delta f(x) = f(qx) - f(x),\ \Delta^i f(x) = \Delta(\Delta^{i-1} f(x)),$$

then, precisely as in the case of the usual finite difference operator, $\Delta^n f(x)$ is expressible* linearly (with constant coefficients) in terms of $f(q^i x)$, $(i=0, 1, \cdots, n)$, and likewise $f(q^n x)$ linearly in terms of $\Delta^i f(x)$, $(i=0, 1, \cdots, n)$. Hence a linear equation† in $\Delta^i f(x)$, $(i=0, 1, \cdots, n)$, may always be written in the form (2), and vice versa.

One may properly say that the q-difference equation (2) in general presents a simpler problem than does the difference equation (1). The notion of asymptotic form, so necessary for the analytic theory of (1), has as yet played no role in that of (2). It may therefore appear a little surprising that the development of the theory of (2) in any general sense preceded by only a short space the beginning of the analytic theory of (1), which marked the reawakening of an interest, long quiescent, in the subject of difference equations.

2. *Periodic Functions.* The equation

$$(4) \qquad \Delta f(x) = 0,$$

remarked by Babbage‡ in 1815, is among the first q-difference equations to appear in the literature. Its solutions are naturally of primary importance in the theory of equation (2), since func-

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* This fact appears to have been observed first by Thomae, *Les séries Heinéennes supérieures, ou* . . . , Annali di Matematica, (2), vol. 4 (1870–71), pp. 105–138.

† Equations of this kind were styled "difference equations" at least as early as 1847; see Heine, *Untersuchungen über die Reihe*

$$1 + \frac{(1-q^\alpha)(1-q^\beta)}{(1-q)(1-q^\gamma)}x + \frac{(1-q^\alpha)(1-q^{\alpha+1})(1-q^\beta)(1-q^{\beta+1})}{(1-q)(1-q^2)(1-q^\gamma)(1-q^{\gamma+1})}x^2 + \cdots,$$

Journal für Mathematik, vol. 34 (1847), pp. 285–328.

‡ Babbage, *An essay towards the calculus of functions*, Royal Society of London, Philosophical Transactions, 1815, Part 2, pp. 389–423.

tions of this character but otherwise arbitrary play for (2) the role of arbitrary constants in the theory of linear differential equations. These "multiplicatively periodic" functions also are of importance for the theory of the generalized hypergeometric series studied by Heine* and Thomae† and of the "higher Heine series,"

$$1+\sum_{n=1}^{\infty}\frac{1-q^{a}}{1-q}\frac{1-q^{a+1}}{1-q^{2}}\cdots\frac{1-q^{a+n-1}}{1-q^{n}}\cdot\frac{1-q^{a'}}{1-q^{b'}}\frac{1-q^{a'+1}}{1-q^{b'+1}}$$

$$\cdots\frac{1-q^{a'+n-1}}{1-q^{b'+n-1}}\cdots\frac{1-q^{a^{(k)}}}{1-q^{b^{(k)}}}\frac{1-q^{a^{(k)}+1}}{1-q^{b^{(k)}+1}} \tag{5}$$

$$\cdots\frac{1-q^{a^{(k)}+n-1}}{1-q^{b^{(k)}+n-1}}x^{n},$$

investigated by Thomae.‡ The properties of the solutions of (4) were made the object of a rather thorough study by Pincherle§ in 1880; these functions were also treated by Rausenberger‖ in 1884. In the following pages the term "periodic function" is to be understood as meaning a solution of (4).

3. *q-Finite Integration.* The problem of q-finite integration, that is, of solving the equation $\Delta f(x) = b(x)$, was examined by Goursat¶ in 1903–4 in connection with his investigation of a problem in the theory of partial differential equations. Goursat assumed $b(x)$ analytic for $|x|<R$ and determined, by the method of power series with direct convergence proofs, under what conditions there exists a solution of like character. He

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* See the second footnote on p. 363.

† Thomae, *Beiträge zur Theorie der durch die Heinesche Reihe:*

$$1+\frac{1-q^{a}}{1-q}\cdot\frac{1-q^{b}}{1-q^{c}}x+\frac{1-q^{a}}{1-q}\cdot\frac{1-q^{a+1}}{1-q^{2}}\cdot\frac{1-q^{b}}{1-q^{c}}\cdot\frac{1-q^{b+1}}{1-q^{c+1}}x^{2}+\cdots$$

darstellbaren Functionen, Journal für Mathematik, vol. 70 (1869), pp. 258–281.

‡ See the first footnote on p. 363.

§ See the first footnote on p. 362.

‖ Rausenberger, *Lehrbuch der Theorie der Periodischen Functionen . . .*, 1884, 8+476 pp.; especially pp. 221 ff.

¶ Goursat, *Sur un problème relatif à la théorie des équations aux derivées partielles du second ordre,* Toulouse Annales, (2), vol. 5 (1903), pp. 405–436; vol. 6 (1904), pp. 117–144.

observed that for such a solution to exist $b(0)$ must vanish, and proved that in the case of $b(0)=0$ and $|q|\neq 1$ there exists a solution analytic for $|x|<R$. For $|q|=1$ two cases must be distinguished, according as q is or is not a root of unity. In the first let m be the least integer for which $q^m=1$; then a solution exists if and only if the powers of x^m in the Maclaurin expansion of $b(x)$ all have coefficients zero. Moreover, when a solution exists it is determined only up to an additive arbitrary function of x^m analytic for $|x|<R$. In the second case a solution may or may not exist, but this depends upon the nature of arg q rather than upon $b(x)$. Goursat also gave a supplementary consideration to the problem in the real domain. It may be remarked that definite and indefinite q-finite integrals of certain elementary functions were found by F. H. Jackson* from 1910 to 1917.

Further light will be shed on this problem in a later section.

4. *Homogeneous Equations. Case of* $a_0(0)\neq 0$, $a_n(0)\neq 0$. The equation (2h) of order $n>1$ was at first studied only under very restrictive hypotheses. In 1847 Heine† showed that a particular equation of the second order is satisfied by the "Heine series." It was some years later, in 1870–71, that Thomae‡ went further in proving that the equation (2h) of order n whose coefficients are linear in x and whose characteristic equation has no infinite or zero roots can be solved in terms of the series (5) of order $h=n$. In 1909–11 Jackson§ found solutions for several particular equations (2), mainly of the second order and homogeneous; most of his solutions also were expressed in power series of the generalized hypergeometric type.

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* Jackson, *Borel's integral and q-series*, Proceedings of the Royal Society of Edinburgh, vol. 30 (1910), pp. 378–385; *A q-generalization of Abel's series*, Palermo Rendiconti, vol. 29 (1910), pp. 340–346; *q-difference equations*, American Journal of Mathematics, vol. 32 (1910), pp. 305–314; *On q-definite integrals*, Quarterly Journal of Mathematics, vol. 41 (1910), pp. 193–203; *The q-integral analogous to Borel's integral*, Messenger of Mathematics, vol. 47 (1917), pp. 57–64.

† See the second footnote on p. 363.

‡ See the first footnote on p. 363.

§ Jackson, *Generalization of the differential operative symbol with an extended form of Boole's equation*, Messenger of Mathematics, vol. 38 (1909), pp. 57–61; *q-difference equations*, American Journal of Mathematics, vol. 32 (1910), pp. 305–314; *The products of q-hypergeometric functions*, Messenger of Mathematics, vol. 40 (1911), pp. 92–100.

Let us now assume the coefficient functions $a_i(x)$ in $(2h)$ to be analytic at $x=0$ and hence expressible in the form

$$a_i(x) = a_{i0} + a_{i1}x + a_{i2}x^2 + \cdots \text{ for } |x| < R, \ (i = 0, 1, \cdots, n);$$

the characteristic equation (3) may then be written

$$a_{00}\rho^n + a_{10}\rho^{n-1} + \cdots + a_{n-1,0}\rho + a_{n0} = 0. \tag{6}$$

We distinguish two essentially different cases according as the roots of (6) are or are not all finite and different from zero; for the present we confine ourselves to the former, assuming $a_{00} \neq 0$, $a_{n0} \neq 0$.

Let $\rho_j = q^{r_j} (j = 1, 2, \cdots, p)$ be any set of p roots $(p \geqq 1)$ of equation (6) satisfying the following conditions: (a)

$$r_j = r_1 - m_j, \qquad (j = 2, 3, \cdots, p),$$

where each m_j is a positive integer or zero and the subscripts of the r's are so chosen as to make

$$0 \leqq m_2 \leqq m_3 \leqq \cdots \leqq m_p;$$

(b) no root of (6) outside this set is equal to q^{r_1+m} for m a positive or negative integer or zero. Then there correspond to the roots of this set p formal solutions of $(2h)$:

$$\begin{aligned}
&x^{r_1}P(x), \\
&x^{r_2}[P(x) + tP(x)], \\
&x^{r_3}[P(x) + tP(x) + t^2P(x)], \\
&\quad \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \\
&x^{r_p}[P(x) + tP(x) + t^2P(x) + \cdots + t^{p-1}P(x)],
\end{aligned} \tag{7}$$

where we have set

$$t = \frac{\log x}{\log q}, \tag{8}$$

and where $P(x)$ is used in a generic sense to denote a power series in x. The coefficients in the several power series $P(x)$ can be found immediately by the method of undetermined coefficients, upon substituting the series (7) in $(2h)$. Since all the roots of (6) can clearly be grouped in sets satisfying the conditions (a) and (b), a set of n formal solutions of $(2h)$ is thus obtained.

Each of the power series $P(x)$ converges for x in the vicinity of the origin, and therefore the existence of n solutions of $(2h)$ analytic in this neighborhood follows. The existence and convergence of these formal solutions is an immediate consequence of a theorem of Grévy,* who in 1894, basing his work on certain earlier studies by Königs, investigated a functional equation which includes $(2h)$ as a particular case. It also follows from Grévy's work that no identical linear relation, with analytic coefficients satisfying the equation (4) and not all identically zero, exists between these n solutions; hence they are said to constitute a fundamental set for the equation $(2h)$.

When no root of the characteristic equation (6) is equal to an integral power of q (including q^0) times another, so that none of the formal series solutions contain logarithmic terms, the convergence of the series and the consequent existence of analytic solutions follows also from results obtained in 1912 by Carmichael.† He employed a method of successive approximation analogous to that which he had already used in studying difference equations,‡ and gave the first treatment of the equation $(2h)$ of order n as such with coefficients other than linear in x. He remarked that the method of Birkhoff§ for difference equations can also be applied in the case considered; not only is this true, but the same results can be obtained in this way with considerably less labor. Carmichael also investigated the case of $|q|=1$, showing that in general analytic solutions do not exist and examining their nature when they do exist.

In 1915, T. E. Mason‖ proved that if $|q|$ is >1 and the coefficients $a_i(x)$ are entire functions, with $a_0(x)\equiv 1$, and if the roots of (6) are not all zero, there exist one or more solutions of the form $f(x)=x^{r_i}E_i(x)$, where $E_i(x)$ is entire. This result was obtained by reckoning out the coefficients in the formal series

* Grévy, *Étude sur les équations fonctionnelles*, (Dissertation, Paris), 1894; reprinted in large part in Annales de l'École Normale Supérieure, (3), vol. 11 (1894), pp. 249–323.

† See the second footnote on p. 362.

‡ Carmichael, *Linear difference equations and their analytic solutions*, Transactions of this Society, vol. 12 (1911), pp. 99–134.

§ Birkhoff, *General theory of linear difference equations*, Transactions of this Society, vol. 12 (1911), pp. 243–284.

‖ Mason, *On properties of linear q-difference equations with entire function coefficients*, American Journal of Mathematics, vol. 37 (1915), pp. 439–444.

solutions and proving their convergence directly, and it was shown that the order of the entire function $E_i(x)$ is not greater than the maximum order of the coefficients.

Direct proof of the convergence of the formal series (7) under all conditions was given by Adams* in 1929. This method appears to have some advantage in brevity and simplicity over the others which have been used for establishing the existence of analytic solutions. It may be remarked that Mason's results can now immediately be extended to show that if the coefficients are entire functions, with $a_0(x) \equiv 1$ and $|q| > 1$, each of the power series in (7) converges for all finite values of x and represents an entire function whose order is not greater than the maximum order of the coefficients.

For brevity let us assume for the moment that $|q|$ is >1. Then if the coefficients $a_i(x)$ have only isolated singularities in the finite plane and $a_0(x)$ vanishes only at isolated points, it is clear that the solutions can be analytically continued away from $x=0$ indefinitely by repeated use of the equation (2h) itself. Thus it appears that the nearest singularity of any solution to the origin is no nearer than the nearest singularity of the functions

$$\frac{a_i(x)}{a_0(x)}, \qquad (i = 1, 2, \cdots, n), \tag{9}$$

and hence the radius of convergence of each power series in the set of formal solutions is at least as great as the distance to the nearest singularity of the functions (9). Moreover the singularities are isolated (occurring at no points other than $q^m x_i$ where m is a positive integer and x_i is any singularity of one or more of the functions (9)) and if the functions (9) have no singularities other than poles, as is the case when the $a_i(x)$ are rational functions, then the solutions will be analytic except for poles in the finite plane away from $x=0$. Similar statements hold when $|q|$ is <1, except that $a_0(x)$ is replaced by $a_n(x)$.

If the functions $a_i(x)$ are analytic both at $x=0$ and $x=\infty$ and if the roots of the characteristic equations for both points satisfy the restriction imposed by Carmichael, the equation (2h) has two sets of fundamental solutions, one of the form $x^{r_i} A_i(x)$ with

* Adams, *On the linear ordinary q-difference equation*, Annals of Mathematics, (2), vol. 30 (1929), pp. 195–205.

$A_i(x)$ analytic at $x=0$ and the other of like form with $A_i(x)$ analytic at $x=\infty$. Each set of solutions may then be expressed linearly in terms of the other with coefficients $P_{ij}(x)$ satisfying the equation (4). Employing a system of n equations of the first order rather than a single equation $(2h)$ of order n, Carmichael* examined these periodic functions and showed that those by means of which the solutions associated with $x=0[\infty]$ are expressed in terms of the solutions associated with $x=\infty[0]$ for $|q|>1[<1]$ are analytic away from the points 0 and ∞ when the coefficient functions are polynomials in x and analytic save for poles when the coefficients are rational functions of x.

Necessary and sufficient conditions that the equation $(2h)$ of first order have a rational solution were found by Mason† in 1914. He also showed that the solutions of the equation

$$f(qx) - (1 + x)f(x) = 0$$

are transcendentally transcendental.

In 1918 Carmichael‡ discussed what he termed "repeated solutions" of a linear homogeneous equation involving a general operator D which includes the differential operator, the ordinary difference operator, and the q-difference operator as particular cases. By definition $f(x)$ is said to be an r-fold solution when the functions $x^i f(x)(i=0, 1, \cdots, r-1)$ all satisfy the equation but $x^r f(x)$ does not.§ Carmichael's chief result for q-difference equations is that a necessary and sufficient condition that a function $f(x)$ be a "repeated solution" of a linear homogeneous q-difference equation of order n (expressed in terms of $\Delta^i f(x)$, $(i=0. 1, \cdots, n)$) is that it satisfy both the equation itself and the equation obtained from it by formal differentiation with respect to Δ and the replacement of x by x/q in the coefficient functions.

* See the second footnote on p. 362.

† Mason, *Character of the solutions of certain functional equations*, American Journal of Mathematics, vol. 36 (1914), pp. 419–440.

‡ Carmichael, *Repeated solutions of a certain class of linear functional equations*, Tôhoku Mathematical Journal, vol. 13 (1918), pp. 304–313.

§ That this terminology has a reasonable basis when D represents the differential or difference operator is manifest, but it seems inappropriate when D stands for the q-difference operator, since an equation $(2h)$ with constant coefficients having $f(x)$ as a particular solution corresponding to a multiple root of the characteristic equation has $(\log x)f(x)$ as a second solution rather than $xf(x)$.

He also obtained such a condition in terms of the coefficients alone.

An existence proof quite different from those cited above was given in 1921 by Ryde,* who employed the "geometric factorial series"†

$$(10) \qquad a_0 + \sum_{s=1}^{\infty} \frac{a_s}{(1-x)(1-x/q)\cdots(1-x/q^{s-1})}.$$

He showed first that a necessary and sufficient condition that a given function be developable in a series of this type is that this function be analytic at $x=\infty$. The coefficients in $(2h)$ were then assumed to be expressed in terms of series of type (10) and it was proved that the equation is formally satisfied by series of the same kind, multiplied by suitable Heinean functions. The existence of analytic solutions was shown by proving the convergence of the formal series.

In 1922 Carmichael‡ indicated how the study of systems of algebraic equations may be used as a guide in determining the properties of solutions of q-difference equations, ordinary and partial, and of integro-q-difference equations, to which one may pass from the algebraic systems by suitably chosen limiting processes. The independent variable x is assumed to be real and the problems of oscillation, comparison, and expansion are the ones specially considered. A somewhat detailed treatment of these problems, namely, oscillation properties of a fundamental system of solutions of $(2h)$ of order 2 and of order $n>2$, comparison properties for the solutions of two particular equations $(2h)$ of order 2, and expansion properties for adjoint systems of order n containing a parameter linearly—was made in a second paper by the same author.§

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* See the third footnote on p. 362.

† This follows the lines of Nörlund, *Sur l'intégration des équations linéaires aux différences finies par des séries de facultés*, Palermo Rendiconti, vol. 35 (1913), pp. 177–216.

‡ Carmichael, *Algebraic guides to transcendental problems*, this Bulletin, vol. 28 (1922), pp. 179–210.

§ Carmichael, *Boundary value and expansion problems: oscillation, comparison, and expansion theorems*, American Journal of Mathematics, vol. 44 (1922), pp. 129–152. See also the third footnote on p. 361.

5. *Generalized Riemann Problem.* In 1913 Birkhoff* formulated and solved what may be called the generalized Riemann problem for linear q-difference equations. He dealt with a system of n equations of the first order rather than a single equation of the nth order, and assumed that all the formal series solutions are free from logarithms. As a preliminary step he completely determined the periodic functions by means of which the solutions associated with $x=0$ are expressed in terms of those associated with $x=\infty$ when the coefficients are polynomials. Denoting the respective matrices of solutions by

$$F_0(x) = (x^{\rho_i} f_{ij}(x)), \qquad F_\infty(x) = q^{(\mu/2)(t^2-t)}(x^{-\sigma_i} g_{ij}(x)),$$

where μ denotes the maximum degree of the polynomial coefficients of the system and t is given by (8), and setting

$$F_0(x) = F_\infty(x)P(x), \quad P(x) = (p_{ij}(x)),$$

he showed that $p_{ij}(x)$ has the form

$$c_{ij}\left(\exp\cdot\left\{-\frac{\eta\mu}{2}t^2 + \left[\eta(\sigma_i+\rho_j) - \frac{\eta'\mu}{2}\right]t\right\}\right)\sigma(t-\alpha_1^{(i,j)}) \cdots \sigma(t-\alpha_\mu^{(i,j)}),$$

where $\sigma(t)$ is the Weierstrass sigma function associated with the periods 1 and $2\pi(-1)^{1/2}/\log q$ and

$$\sum_{\lambda=1}^{\mu}\alpha_\lambda^{(i,j)} = \sigma_i + \rho_j - \mu\pi(-1)^{1/2}/\log q. \tag{11}$$

The $2n+n^2(\mu+1)$ numbers

$$\rho_j,\ \sigma_j,\ c_{ij},\ \alpha_1^{(i,j)}, \cdots, \alpha_\mu^{(i,j)} \qquad (i, j = 1, 2, \cdots, n), \tag{12}$$

of which only $n^2\mu+1$ are independent, were designated "characteristic constants," and it was proved that for any assigned set of constants (12) subject to the condition (11) there exists a system having this set of characteristic constants or one differing from it only in the replacement of σ_j by σ_j+l_j, where the l_j are integers.

* Birkhoff, *The generalized Riemann problem* $\cdots$, Proceedings of the American Academy of Arts and Sciences, vol. 49 (1913), pp. 521–568.

It would be of considerable interest to extend these results to the case in which some of the formal series solutions contain logarithms.

6. *Non-homogeneous Equations.* In 1915 it was shown by Mason* that the equation $(2n)$, with entire function coefficients, $|q|>1$, and $a_0(x)\equiv 1$, has an entire function solution whenever it has a formal power series solution, and that the order of the solution does not exceed the maximum order of the coefficients.

The first study of the non-homogeneous q-difference equation under general hypotheses appears to have been made by Adams† in 1925. He employed a system of n equations of the first order in preference to the equation $(2n)$ and showed that in a large class of cases, when the coefficients have the character of rational functions at $x=0$, the system is satisfied formally by a series which converges and so represents an analytic solution. The method employed makes use of the notion of "variation of the constants" and of an evaluation for the operator $\sum=\Delta^{-1}$ in terms of series. When the coefficients are of such character that the system admits two solutions, one associated with $x=0$ and the other with $x=\infty$, the relation between the solutions was studied.

The equation $(2n)$ itself was investigated by the same author‡ in 1929 on the assumption that the coefficients $a_i(x)$ and $b(x)$ are all analytic at $x=0$. Here again the existence of analytic solutions was shown by exhibiting series that formally satisfy the equation and proving their convergence directly. These results may be generalized immediately to the case in which $b(x)$ in $(2n)$ has the form

$$b(x) = \sum_{j=0}^{m} t^j b_j(x^{1/s}), \tag{13}$$

where t is given by (8), s is a positive integer $\geqq 1$, and each function $b_j(x^{1/s})$ is an analytic function of $x^{1/s}$ at $x=0$. For our later purposes it will be convenient to have the following statement of this generalization.

* See the fifth footnote on p. 367.

† Adams, *Note on the existence of analytic solutions of non-homogeneous linear q-difference equations, ordinary and partial*, Annals of Mathematics, (2), vol. 27 (1925), pp. 73–83; vol. 30 (1929), p. 626.

‡ See the footnote on p. 368.

If among the quantities q^i, $(i=0, 1, 2, \cdots)$, there are exactly p roots $(p \geqq 0)$ of the equation (6), each root being counted according to its multiplicity, the equation $(2n)$ with $b(x)$ given by (13) has a formal solution

$$(14) \qquad s(x) = P(x^{1/s}) + tP(x^{1/s}) + \cdots + t^{m+p}P(x^{1/s}),$$

in which the coefficients in the power series $P(x^{1/s})$ may be calculated by substitution in the equation $(2n)$. If $|q|$ is $>1[<1]$ and the equation (6) has no infinite [zero] roots, the series in (14) converge in the vicinity of $x=0$.

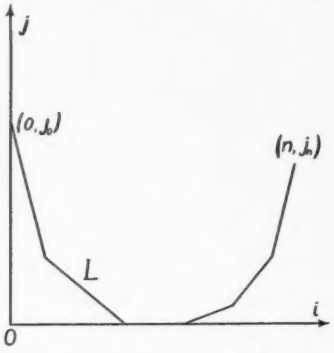
It is of importance to observe that this result gives a q-finite integral of every function of the type (13). One should note also that Mason's results* may immediately be generalized as follows: if $|q|$ is >1 and the coefficients $a_i(x)$ of $(2n)$ are entire functions with $a_0(x) \equiv 1$, and if $b(x)$ is of the form (13) with each $b_j(x^{1/s})$ an entire function of $x^{1/s}$, then each $P(x^{1/s})$ in (14) converges for every finite value of $x^{1/s}$ and represents an entire function of $x^{1/s}$ whose order does not exceed the maximum order of the $a_i(x)$ and $b_j(x^{1/s})$.

7. *Homogeneous Equations. Case of* $a_0(0)=0$ *or* $a_n(0)=0$ *or Both.* In this case also we assume each coefficient $a_i(x)$ to be analytic at $x=0$ and so expressible in the form $\sum_{j=0}^{\infty} a_{ij}x^j$ for $|x|<R$. Let a_{i,j_i} denote the first non-zero coefficient in $a_i(x)$ and plot the points (i, j_i) in the ij-plane. Then construct a broken line L, convex downward, such that both ends of each segment are points of the set (i, j_i) while all points of the set lie on or above this line. No restriction is imposed by assuming at least one point of the set to be on the i-axis, for otherwise an integral power of x could be suppressed from the entire equation.

The equation (6) is now by no means completely characteristic of the difference equation $(2h)$. In fact one might say that (3) is replaced by several characteristic equations, one associated with each segment of L. The degree of the characteristic equation associated with any segment of L is one less than the number of points that lie on or vertically above that segment, including its end points. The coefficients of this characteristic equation are the a_{i,j_i} corresponding to points (i, j_i) actually on that segment; the coefficient corresponding to a point (i, j_i) ver-

* See the fifth footnote on p. 367.

tically above the segment is zero. Evidently the sum of the degrees of these several characteristic equations is n. If one of the segments is horizontal, the characteristic equation associated with that segment may be obtained from (6) by suppressing the



infinite and zero roots. To the roots of this equation there correspond formal solutions of the type (7), in number equal to the degree of the equation. In order to obtain solutions associated with a segment of slope $\mu \neq 0$ we make the transformation

$$f(x) = q^{(\mu/2)(t^2-t)}g(x). \tag{15}$$

This changes the equation ($2h$) into a new equation which, after the suppression of factors common to all its terms, is of exactly the same type except in the respect that if μ is not an integer but a fraction, r/s in lowest terms and s positive, certain of the power series coefficients are multiplied by positive integral powers of $x^{1/s}$. The effect of (15) upon the points (i, j_i) is to relocate them in such a way that each segment of the new broken line L', similarly constructed for the transformed equation, has a slope μ less than that of the corresponding segment of L. Thus, in particular, the segment of L whose slope is μ becomes a horizontal segment of L'.

When μ is an integer there corresponds to the segment of L having this slope a set of formal solutions exactly like (7) except for the additional factor $q^{(\mu/2)(t^2-t)}$ in each one. To a segment whose slope is a fraction, r/s, there corresponds a set of formal solutions like (7) in type except that each contains the addi-

tional factor $q^{(\mu/2)(i^2-i)}$ and each $P(x)$ is replaced by a series in powers of the sth root of x, $P(x^{1/s})$.

A full set of n formal solutions is thus obtained for the equation $(2h)$. These were exhibited by Adams* in 1929, together with direct proof of the convergence, in the vicinity of $x=0$, of the series associated with the segment of L farthest to the left [right] when $|q|$ is >1 [<1]. The type of convergence proof used is clearly not applicable to the series associated with other segments of L; indeed examples in which the series associated with other segments converge only for $x=0$ are easily constructed. Thus the question of whether there always exist analytic solutions associated with the other segments was left open. Some light, however, is shed on the situation by the following discussion, in which we assume for brevity of statement that $|q|$ is >1.

It is clearly no real restriction to suppose that the segment farthest to the left in the broken line L belonging to the equation $(2h)$ is horizontal. The equation then has at least one formal solution free from logarithms,

$$s_1(x) = x^{r_1}(s_1 + s_1' x^{1/s} + s_1'' x^{2/s} + \cdots), \quad q^{r_1} = \rho_1 \neq 0,$$

where s is an integer $\geqq 1$, and this series converges for $x^{1/s}$ in the neighborhood of $x=0$. We propose to employ this solution to reduce the order of $(2h)$; to this end let us set

$$f(x) = s_1(x) \sum g(x), \quad \sum = \Delta^{-1}.$$

Substituting in $(2h)$ we find

$$a_0(x)s_1(q^n x)g(q^{n-1}x) + \left[\sum_{i=0}^{1} a_i(x)s_1(q^{n-i}x)\right]g(q^{n-2}x) + \cdots + \left[\sum_{i=0}^{n-1} a_i(x)s_1(q^{n-i}x)\right]g(x) = 0, \tag{16}$$

for which the equation analogous to (6) is

$$a_{00}\rho_1^n \sigma^{n-1} + \left[\sum_{i=0}^{1} a_{i0}\rho_1^{n-i}\right]\sigma^{n-2} + \cdots + \left[\sum_{i=0}^{n-1} a_{i0}\rho_1^{n-i}\right] = 0. \tag{17}$$

Let us now assume for the moment that the equation (6) has

* See the first footnote on p. 368.

exactly m roots $(0 \leqq m \leqq n-2)$ equal to zero, so that it may be written

$$a_{00}\rho^n + a_{10}\rho^{n-1} + \cdots + a_{n-m,0}\rho^m = 0, \quad a_{n-m} \neq 0.$$

The coefficient of σ^m in (17) is then precisely

$$-\, a_{n-m,0}\rho_1^m \neq 0,$$

while the subsequent coefficients are all zero; thus the equation (17) has exactly m zero roots and consequently one less non-zero root than (6). Moreover if σ_1 is any non-zero root of (17), $\rho_1\sigma_1$ is a root of (6). For we may write (17) for $\sigma = \sigma_1$ and multiply by σ_1 to obtain

$$(18)\quad a_{00}\rho_1^n\sigma_1^n + a_{10}\rho_1^{n-1}\sigma_1^{n-1} + a_{20}\rho_1^{n-2}\sigma_1^{n-2} + \cdots + a_{n-1,0}\rho_1\sigma_1$$
$$+ a_{00}\rho_1^n\sigma_1^{n-1} + \left[\sum_{i=0}^{1} a_{i0}\rho_1^{n-i}\right]\sigma_1^{n-2} + \cdots + \left[\sum_{i=0}^{n-2} a_{i0}\rho_1^{n-i}\right]\sigma_1 = 0.$$

Since σ_1 satisfies (17) the sum of terms in the second line equals

$$-\sum_{i=0}^{n-1} a_{i0}\rho_1^{n-i},$$

which, ρ_1 being a root of (6), is equal to a_{n0}. Thus (18) expresses the fact that $\rho_1\sigma_1$ is a root of (6).

The equation (16) has at least one formal solution

$$s_2(x) = x^{r_2}(s_2 + s_2' x^{1/s} + s_2'' x^{2/s} + \cdots), \quad q^{r_2} = \sigma_1 \neq 0,$$

and the series converges for $x^{1/s}$ in the neighborhood of $x = 0$. From §6 it follows that a q-finite integral of $s_2(x)$ is

$$\sum s_2(x) = x^{r_2}[P(x^{1/s}) + tP(x^{1/s}) + \cdots + t^m P(x^{1/s})],$$

each $P(x^{1/s})$ being convergent in the vicinity of $x = 0$ and m an integer $\geqq 0$. Hence

$$s_1(x) \sum s_2(x)$$

is a solution of $(2h)$, and since $q^{r_1+r_2} = \rho_1\sigma_1$ is a root of (6) this solution must be essentially the same as a solution already known to exist. This process of reducing the order can evidently be carried out repeatedly and a complete set of solutions associated with the segment of L farthest to the left can thus be ob-

tained. If L consists of more than one segment, the reduced equation, after a finite number of reductions, will have a characteristic equation of type (6) all of whose roots are zero. For a suitable value of μ the reduced equation will have at least one solution expressible in the form

$$(19) \qquad q^{(\mu/2)(t^2-t)}x^{r_3}(s_3 + s_3' x^{1/s} + s_3'' x^{2/s} + \cdots)$$

for $x^{1/s}$ in the vicinity of $x=0$. This method of obtaining solutions of $(2h)$ therefore must stop at this point unless a q-finite integral of the function (19) can be determined. Indeed we can proceed further if and only if we can "integrate" functions of the following character,

$$(20) \qquad q^{(\mu/2)(t^2-t)}x^r t^m a(x^{1/s}),$$

$a(x^{1/s})$ being an analytic function of $x^{1/s}$ in the vicinity of $x=0$ and m an integer $\geqq 0$.

If the coefficient functions $a_i(x)$ are meromorphic over the entire finite plane, the function $a(x^{1/s})$ will be a meromorphic function of $x^{1/s}$ in the same region. In this case we can determine a q-finite integral of (20), or in other words a solution of the equation

$$(21) \qquad h(qx) - h(x) = q^{(\mu/2)(t^2-t)}x^r t^m a(x^{1/s}),$$

as follows. Let the variables be changed by setting

$$x = q^z, \quad h(q^z) = \phi(z);$$

then (21) becomes the difference equation

$$(22) \quad \phi(z+1) - \phi(z) = q^{(\mu/2)(z^2-z)}q^{rz}z^m a(q^{z/s})$$
$$= e^{(\mu/2)(z^2-z)\log q + rz\log q} z^m a\left(e^{(z/s)\log q}\right).$$

The first two factors on the right are entire functions; the third has no singularities in the finite z-plane other than poles. Hence the right-hand member of (22) is meromorphic over the entire finite z-plane and there exists* a meromorphic sum of this function, that is, a meromorphic solution of the equation (22).

When a full set of n solutions of the equation $(2h)$ can be obtained by the method indicated, it would be of interest to show

* See Hurwitz, *Sur l'intégrale finie d'une fonction entière*, Acta Mathematica, vol. 20 (1897), pp. 285–312.

that these solutions constitute a fundamental set and to exhibit the relation, if one exists, between the solutions associated with segments of L other than that farthest to the left and the formal series solutions associated with those segments. It would also be of value to discuss the generalized Riemann problem in this case and to investigate the properties of the solutions when the coefficients $a_i(x)$ are entire functions.

8. *Expansion Problems.* A detailed study of expansion problems connected with solutions of the equation (2h) was made by Carman* in 1926. He assumed all the coefficients $a_i(x)$ analytic at $x=0$ and introduced a parameter λ_m linearly into $a_n(x)$. It was shown first that for each integer $m \geqq 0$, λ_m can be determined so that the equation will be satisfied by a function

$$u_m(x) = x^{m+\nu}\left(1 + \sum_{j=1}^{\infty} c_{mj}x^j\right),$$

where ν is any constant and the series converges in the vicinity of $x=0$. Then followed a demonstration that x^p (p an integer $\geqq 0$), and likewise any given function analytic at $x=0$, can be expanded in a series

$$\sum_{m=0}^{\infty} c_m U_m(x), \qquad U_m(x) = x^{-\nu}u_m(x),$$

uniformly convergent in a circle about $x=0$. Carman established a similar expansion theorem for a function analytic at $x=\infty$, developed an analog of the Laurent series, and proved a number of orthogonality conditions for the sets of functions $U_m(x)$ and $u_m(x)$, the integrals being taken over a closed path in the vicinity of $x=0$ encircling that point once. Generalizations of these results for functions of several variables were also pointed out.

9. *Partial q-Difference Equations.* Linear partial q-difference equations have been studied by Adams in four papers from 1924 to 1929. In the first of these† the equation

* Carman, *Expansion problems in connection with homogeneous linear q-difference equations*, Transactions of this Society, vol. 28 (1926), pp. 523–535.

† Adams, *The general theory of a class of linear partial q-difference equations*, Transactions of this Society, vol. 26 (1924), pp. 283–312; the contents of this paper, as well as that referred to in the fourth footnote on p. 379, are contained

$$\sum_{i=0}^{n} a_i(x, y) f(q^{n-i}x, r^{n-i}y) = 0 \tag{23}$$

with coefficients $a_i(x, y)$ analytic at (0, 0) was considered. Solutions were found by obtaining them for the related system of n equations of the first order, following the method of Birkhoff* already cited. When the system possesses n sets of formal solutions in powers of x and y (as will be the case when the characteristic equation for (0, 0) has no zero or infinite roots and no root equal to another times $q^a r^b$ for integral a and b) and when both $|q|$ and $|r|$ are greater or less than 1, these series converge in the vicinity of (0, 0) and represent analytic solutions there. The equation of first order whose characteristic equation has the root zero was also examined in this paper. It was shown that for $|q| = |r| = 1$ the equation in general admits no analytic solution that is not identically zero. When the coefficients $a_i(x, y)$ are rational functions, the fundamental periodic functions by means of which one set of solutions is expressed in terms of another (for example, the solutions associated with (0, 0) and with (∞, ∞) when both $|q|$ and $|r|$ are greater or less than 1) are analytic away from $x=0$, $x=\infty$, $y=0$, $y=\infty$; moreover these functions are triply periodic and of the class studied by Cousin† in 1910.

In the second paper‡ Adams proved for the non-homogeneous equation of type (23) results analogous to those already cited for the equation (2n). The third paper§ was devoted to the mixed equation

$$f(x + 1, ry) = a(x, y)f(x, y),$$

where $a(x, y)$ is a polynomial with constant term not zero. The existence of principal solutions was established by the method of Birkhoff,‖ and the relation between two such, when they exist, was examined.

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in essentially the same form in *The general theory of the linear partial q-difference equation and of the linear partial difference equation of the intermediate type*, (Dissertation, Harvard), 1922.

* See the fourth footnote on p. 367.

† Cousin, *Sur les fonctions triplement périodiques de deux variables*, Acta Mathematica, vol. 33 (1910), pp. 105–232.

‡ See the second footnote on p. 372.

§ Adams, *Existence theorems for a linear partial difference equation of the intermediate type*, Transactions of this Society, vol. 28 (1926), pp. 119–128.

‖ See the fourth footnote on p. 367.

Adams' fourth paper* treated the equation

$$\sum_{i,j=0}^{n} a_{ij}(x, y) f(q^{n-i}x, r^{n-j}y) = 0, \tag{24}$$

with coefficients analytic at (0, 0). If we let

$$a_{ij}(0, 0) = a_{ij00},$$

the characteristic equation of (24) at (0, 0) is

$$\sum_{i,j=0}^{n} a_{ij00}\rho^i\sigma^j = 0.$$

A pair of values (ρ_0, σ_0) satisfying this equation is called a characteristic number-pair. It was demonstrated that under general conditions there corresponds to such a number-pair a formal solution which in the general case is of the form

$$x^\mu y^\nu [(P.S.) + t(P.S.) + t^2(P.S.) + \cdots + t^m(P.S.)],$$

where $\mu = \log \rho_0/\log q$, $\nu = \log \sigma_0/\log r$, and where $(P.S.)$ stands in a generic sense for a power series in x and y, and t represents one or the other of the functions

$$\frac{\log x}{\log q}, \quad \frac{\log y}{\log r}.$$

The existence of analytic solutions was established by direct proof of the convergence of these formal series. The non-homogeneous equation of type (24) was also considered.

In connection with this last paper several interesting questions arise. Under certain conditions there appear to be a smaller number of formal solutions associated with a characteristic number-pair than one would expect; are there others which have not yet been exhibited? Sometimes also there appear to be a larger number than would normally be anticipated; are these in a proper sense linearly dependent? What is the most general solution of an equation of type (24)?

10. *Differential q-Difference Equations.* The only detailed consideration yet given to the differential-q-difference equation

* Adams, *On the linear partial q-difference equation of general type*, Transactions of this Society, vol. 31 (1929), pp. 360–371.

is by Flamant* who in 1924 made an elaborate study of the equation

$$f'(x) = a(x)f(qx) + b(x), \quad (|q| < 1), \tag{25}$$

where $a(x)$ and $b(x)$ are analytic in a closed and simply connected region D such that if x is in D, qx is likewise. It is clear that the combination of conditions placed upon $f(x)$ by (25) and by a prescribed value at $x=0$ may be expressed by an integral equation of Volterra type. By a method of successive approximation the author established the existence of a unique solution analytic in D. He indicated how the results may be generalized to a type of equation which includes as a particular case the equation obtained from (25) by replacing $a(x)f(qx)$ by a linear combination of the functions $f(q^ix)(i=1, 2, \cdots, n)$ with coefficients analytic in D. The greater portion of the paper was devoted to examining the effect upon the solutions of permitting $b(x)$ to have singularities in D. For example, a pole of order $\mu>1\,[=1]$ of $b(x)$ at $x=x_0\neq 0$ impresses upon the solution of (25) poles of order $\mu-1$ [logarithmic singularities] at the points x_0/q^n of the region D. When logarithmic singularities are present it is possible to develop the solution in a series involving logarithms. The case of $|q|=1$ was also given detailed treatment, as well as that in which $b(x)$ has singularities of various types at $x=0$.

11. *Integro-q-Difference Equations.* Integro-functional equations have been the object of study by several authors. Some of these equations, especially the one examined by Tamarkin,† include as a particular case an equation in which the unknown function $f(x)$ is subjected both to an integral operator of Volterra type and to the q-difference operator of first order. In 1929 Adams‡ considered the equations obtained from $(2h)$ and $(2n)$ by adding to the right-hand member the term

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* Flamant, *Sur une équation différentielle fonctionnelle linéaire*, Palermo Rendiconti, vol. 48 (1924), pp. 135–208.

† Tamarkin, *On Volterra's integro-functional equation*, Transactions of this Society, vol. 28 (1926), pp. 426–431.

‡ Adams, *Note on integro-q-difference equations*, Transactions of this Society, vol. 31 (1929), pp. 861–867.

$$\int_0^x K(x,\xi)f(\xi)d\xi.$$

When the known functions involved are analytic at $x=0$, $\xi=0$, these equations are in general satisfied formally by certain power series, multiplied by suitable powers of x. If $|q|$ is >1 the convergence of these series can be established and solutions analytic in the vicinity of $x=0$ are thereby determined. If $|q|$ is <1 the situation is essentially different. Under further restrictions an immediate extension of Tamarkin's results establishes the existence of a unique solution analytic in the neighborhood of $x=0$ when $b(x)$ does not vanish identically, and shows that when $b(x)$ is identically zero there exists no solution bounded in the neighborhood of $x=0$. Under a different set of further restrictions, in some respects less stringent, the existence of a solution analytic at $x=0$ is demonstrated by a method of successive approximation. Special consideration is also given in this paper to the problem in the real domain. A question of interest, as yet unanswered, is: what is the most general solution of such an equation?

12. *Other Problems.* In 1930 Starcher* showed how the equation

$$xf(qx) - f(x) = 0, \qquad (|q| < 1),$$

may be made the basis for the derivation of the two fundamental expressions for the θ-functions and for the deduction of their properties. In a forthcoming paper† this author also obtains a large number of identities by equating different expressions for the solution of a q-difference equation, of the first order nonhomogeneous or of the second order homogeneous, with linear coefficients. Some of these identities are new; others, already given by Euler, Gauss, Cauchy, and other more recent writers, are established by a new method. Number-theoretical interpretations are given to some of the results.

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* Starcher, *A solution of a simple functional equation as a basis for readily obtaining certain fundamental formulas in the theory of elliptic functions*, this Bulletin, vol. 36 (1930), pp. 577–581.

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SUPPLEMENTARY TO NÖRLUND'S BIBLIOGRAPHY OF THE CALCULUS OF FINITE DIFFERENCES AND DIFFERENCE EQUATIONS

Nörlund's *Differenzenrechnung* (1924) includes a notable bibliography, the first of any considerable importance in this field. On account of the rapid growth of the literature of difference equations and related subjects we venture to present here a list supplementing that of Nörlund. Of the following titles, numbering well over three hundred, something over ninety come within the period covered by Nörlund; the rest are of more recent publication. Except for the inclusion of a few book reviews and of a small number of abstracts of papers presented to the American Mathematical Society but apparently not yet published in full, it has been our intention so far as possible to employ the same criteria for acceptance of titles that Nörlund used. Only five of the titles have not been seen by us; these are designated by asterisks.

It may be observed that Nörlund's title, F. H. Jackson [14], should have been listed as C. S. Jackson [1]. In general, however, Nörlund's bibliography is exceedingly accurate and complete, and we hope that ours may be found a worthy and useful supplement to it.*

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* The writer wishes to express his grateful appreciation to R. C. Archibald for many valuable suggestions concerning the preparation of this list.

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REMINISCENCES OF GIBBS BY A STUDENT AND COLLEAGUE*

BY EDWIN BIDWELL WILSON

When last spring I was asked to agree to give the Gibbs Lecture on this occasion, I assented on the assumption that before the time came for the lecture to be delivered I should have been entirely relieved of my executive responsibilities to the Social Science Research Council and should have had opportunity to prepare from a considerable dossier of investigations in progress a scientific paper somewhat fit to take its place with the previous lectures given in this series. As it has turned out, I have not had in full the expected relief and must appear before you with a very general talk of a personal and reminiscent nature. I will not apologize; it may well be that you prefer that type of address from me, and in view of its type I must be expected to speak without apology a good deal in the first person. After all one's personal recollections are his own; they may have little of truth in them, for memory is not infallible; to use a detached third-person style of composition may give them in appearance a greater substance of objective fact than they really merit.

To give you some appreciation of the very inadequate background with which, at the early age of twenty, I came into contact with J. Willard Gibbs, may I state that my undergraduate work was at Harvard and in mathematics, which meant pure mathematics. In the spring of 1899, Professor W. F. Osgood, with whom I had taken a number of courses and who was good enough to take a real and much appreciated interest in me, suggested that I go to Yale for my graduate work. Some of you who have a knowledge of the relative standing in pure mathematics of the departments at Harvard and Yale at that time may think the advice extraordinary. It was, but it was extraordinarily good. As Professor Osgood pointed out, I had been long enough at Harvard and had specialized sufficiently in mathematics to

* The eighth Josiah Willard Gibbs Lecture, delivered, by invitation of the Council of the Society, before the Society and the American Association for the Advancement of Science, December 30, 1930.

get the greater part of the best Harvard had to offer in point of view, and a change would be beneficial to me.* He spoke of Pierpont and of Percey Smith whose interests were somewhat different from his own and those of Professor Bôcher with whom I had had more work than with any other than Osgood. It is my impression that neither Osgood nor Bôcher mentioned Gibbs to me. But when B. O. Peirce heard that I had decided to go to Yale he remarked that down there I might come across Gibbs "whom some of us here think a rather able fellow." Had I known Peirce then as well as I came to know him later I should have taken this remark as indicating a person of the highest quality under whom I must surely plan to study, but at the time I disregarded it entirely. I went to Yale to study with Pierpont and Smith.

How came it that I studied with Gibbs? That was one of life's minor tragedies. When I got to Yale in the autumn of 1899 and was laying out my year's work with A. W. Phillips, Dean of the Graduate School, it appeared that there were only three courses I considered worth while, whereas four were needed for full time work. Phillips suggested that I add Gibbs' Vector Analysis. I protested that according to its description it was not materially different from quaternions, of which I had had a full year under J. M. Peirce, and should hardly count as a course for me. The logic was unanswerable but the circumstances overbore it; I had to have four courses and the Dean would count Vector Analysis even though it was a sort of review; so I registered for it with a sneaking suspicion that my good master Osgood had made a mistake in sending me from a mathematically first rate institution to a second rate one. It was one of life's minor tragedies, but too late to be helped.

You are doubtless impatient that I should get along to talking about Gibbs and anxious for me to quit telling of myself; but I am just as anxious that you should realize what sort of person I was when I reached Yale after being graduated at the age of barely twenty at the head of my class with highest final honors

* This sort of generosity is not unusual at Harvard; taken with reasonably good provision for traveling fellowships, it has deprived the Harvard Graduate School of a goodly number of students of the best grade, much to the advantage of the students and of science, and thus indirectly to the advantage of the University.

in mathematics. I was certainly immature. I was not wise enough to be confident that a new place, new contacts, new points of view have sure advantages which overbear many a technical disadvantage. I was not wise enough to know that to take a subject twice from different angles and thus better master the whole might be far better in the building of a scientific life than to be forever going on to some new subject, leaving everything both new and old with insufficient consolidation. It is not reasonable for you to suppose that, during the brief period from September 1899 when I first saw Gibbs to June 1902 when I took leave of him to go study in Paris, never to see him again, I should have matured very greatly. If I could so have failed in seeing the significance of the remark of B. O. Peirce cited above, it is certain that I must have let slip many things of importance and misinterpreted or falsely remembered many others which occurred during the period of three academic years in which I came in contact with Gibbs. You and I alike are on very insecure ground in believing anything I may recount here today.

The course on Vector Analysis was small; none of Gibbs' courses had more than a mere handful of students, four or six or possibly eight. The course was difficult for everybody in it but me, and was easy for me only because I had previously had quaternions (which incidentally I had found difficult and perplexing though I was amply prepared). The lectures followed the pamphlet which the author had printed privately in 1881–84 but had never published. There were no exercises assigned to the students to work—a truly continental type of course but embarrassing to Americans who are used even in graduate work to having the path made easy for them. The next year, thirty years ago this month, Professor Morris, editor of the Yale Bicentennial Series, asked me to prepare for that Series a text on Vector Analysis and told me that Gibbs had given his consent and that I should talk the matter over with him. The conference was short. Gibbs remarked that he had prepared his pamphlet for the convenience of his students and for distribution privately. He said that he was busy preparing a volume for the same Bicentennial Series (his *Statistical Mechanics*) and would not have time to advise on the composition of the *Vector Analysis*, to read the manuscript or the proof, that I must be entirely responsible for the whole work, that I was free to write whatever kind of

book I pleased, to incorporate so much of his course or pamphlet as I wished and to add whatever I desired from other sources. Somewhat seriously impressed with the magnitude and lonesomeness of the task I said I would do my best, to which he kindly replied that he had confidence that I would do very well, and after the book had appeared he was good enough to remark that it was satisfactory. That is about all the contact I had with him on the *Vector Analysis*.

One topic which he treated at some length, but which I chose to leave out of the book was crystallography; another was the theory of orbits. The latter is adequately represented in his collected works, but the former is nowhere a matter of record and I am sorry that I omitted it, particularly as all my notes on all of Gibbs' courses were lost overboard by careless handling on the part of the crew of the steamship on which I returned from Paris in 1903 so that I had thereafter no way of reconstituting from my notes special material from his courses not found in print. One of the ablest students Gibbs had had in the nineties was G. P. Starkweather, a person as systematic as he was able. He had written out with great care his notes on Gibbs' lectures. After his early death these notes were deposited in the Yale Library. It was chiefly from a volume of these that I was able to put together that part of Gibbs' course on multiple algebra which I found it desirable to print in the Transactions of the Connecticut Academy in 1907 as a precursor to some uses I wished to make of the method in developing some geometrical theorems. I think it safe to say that the treatment of crystallography, though neat and interesting was not of any great importance except as illustrating how the methods of vector analysis could be made to convert the goniometric measures taken on crystals into the desired constants of the crystal.

At one time when Pierpont was lecturing on elliptic functions with some reference to the motion of the top, Gibbs happened to be developing by vectorial methods and discussing the physical meaning of the equations of motion of the top. He turned to the class and with the pleasant smile which often lighted his countenance remarked that there were those who thought the top chiefly interesting as affording an exercise in the use of elliptic functions, but that he found the top a very interesting physical object on its own account. No criticism of another was

implied in the remark, merely an emphasis on his own point of view which he was developing to his class. Although Gibbs had purely mathematical interests as in his Vice-Presidential address on Multiple Algebra to the American Association for the Advancement of Science in 1886 and in his course on the same subject, his abiding interests were in real physical things and he rarely if ever developed his mathematical theories of physics further than was necessary to get at the important physical significance of the phenomena he was discussing; his mathematical methods were the simplest which he could devise and often extended little beyond close logical analysis. That was one reason his courses were hard; technical dexterity is easier than thinking. Read the great thermodynamic memoir if you desire verification of these statements.

Except in the classroom I saw very little of Gibbs. He had a way, toward the end of the afternoon, of taking a stroll about the streets between his study in the old Sloane Laboratory and his home—a little exercise between work and dinner—and one might occasionally come across him at that time. Then there were the meetings of the mathematical and the physical clubs on occasional evenings with papers read by the staff or students. I do not remember that he ever read a paper on such occasions but he was usually in attendance and apparently paying close attention; sometimes he would make very brief remarks after the speaker concluded and the penetration of those comments was noteworthy. On one such occasion when we had been hearing of the then quite new electron theory of the constitution of the atom Gibbs said that it must be getting nearly time for him to move on, that for many years he had been troubled over the problem of reconciling the number of degrees of freedom in the molecules with the value of the ratio of the specific heats at constant pressure and constant volume and that if we were to introduce all the new degrees of freedom implied by the electron constitution he would be still more at a loss. This was, of course, before the introduction of the quantum conditions.

He could be seen at faculty meetings, quiet and attentive. I do not recall hearing him speak but once, and then with few words much to the point. Once I ran across him in the library surrounded by books on the theory of numbers and reading a thesis on algebraic numbers just presented for the doctorate.

I remarked that I had not realized that he was familiar with the theory of algebraic numbers. He replied that he was not, but that with the aid of some books he thought he might be able to come to a decision as to whether the thesis was worthy of acceptance. Once I desired to consult some books which were not in the library but which I had seen on the shelves in his office during a lecture. I ventured to ask whether I might borrow them. He was entirely willing. As I picked the books off the shelves I noticed that the pages had not been cut and enquired whether I might cut them, to which he replied: Certainly, if you think it worth while. Probably I looked abashed, for he added: The author kindly sends me all he writes; there is a great deal of it; I sometimes feel that a person who writes so much must spread his message rather thin.

There may be some interest in a letter written by Gibbs to me just a month before he died:

New Haven Mch 28/03

Dear Mr Wilson

I think that you will have next year
2 hrs Non Euclidean Geometry
2 hrs Mechanics
2 hrs Introduction to Math. Phys.
6 hrs Freshman,
or something very like that. We will know better a little later.

I think that the reasons wh you expressed so eloquently & I may add so discreetly to Dr. B——, would apply to an abridgment. We cannot take for granted that an abridgment wd not interfere with the sale of the larger book. The larger book is pretty heavily handicapped by its price, as it is. & in competition with a cheaper edition could hardly hold its own. Moreover, anything requires time to be well done, & I think to write a short book takes as much time as a longer one.

I did not mean to say that Hamilton did not have the equation

$$q^2 - 2Sq \cdot q + (Tq)^2 = 0.$$

He doubtless would recognize the equation as correct, & may have written it in just that form. Only I do not see how

he could have recognized that it is (I don't care whether you say, identical with or) analogous to the Ham. Cayley equation, because I suppose that he never was aware that a quaternion might be regarded as a matrix. I suppose that that was a discovery of the elder Peirce, as stated by Cayley in the chapter wh he wrote in Tait's *'Quaternions'*.

I am glad that you find that instruction in America compares not too unfavorably with that in France. However, what you want to do is to get the best you can out of France, wh certainly will be a great addition to anything wh you may get here. I am a little surprised that you find the French Lecturers *going to pieces*. I had supposed that that was just what they never did—that they always gave their lectures in good form.

Yours truly
J. Willard Gibbs

Let me interrupt these personal reminiscences to give you a bit of history as it appears in the record. Gibbs was born on February 11, 1839. He prepared for college at the Hopkins Grammar School. He was graduated from Yale in the class of 1858 at the age of nineteen. In college his interests appear to have been Latin and mathematics as he took prizes in each in more than one year of his course. He took the Bristed Scholarship of $95 for the best examination in Greek, Latin and Mathematics. He won the Latin Oration in both Junior and Senior years. He was awarded the Clark Scholarship of $120 for the best examination in the studies of the college course which was conferred subject to the condition that the recipient continue as a graduate for one or two years pursuing non-professional studies. He did so continue and in 1863 got his Ph.D. degree with a thesis: "On the form of the teeth of wheels in spur gearing." In the Yale catalogs of 1863–64 and 1864–65 he is listed as Tutor in Latin; in that of 1865–66 he appears as Tutor in Natural Philosophy. Afterwards he went abroad to study. In the catalog of 1871–72 he reappears as Professor of Mathematical Physics and so continues until his death.

Except for his periods as Tutor he taught only graduate work, although particularly competent undergraduates might be admitted to his courses, especially the Vector Analysis. It may

interest you to follow the subjects he taught. From 1871 to 1881 the topics announced were capillarity, wave theory of light and sound, least squares, and potential theory with applications to electricity and magnetism. It is not to be presumed that he taught all of these subjects in any one year, but the catalogs fail to state just which he did teach. It was in this period that the great papers on thermodynamics were published, but there is no reference to his teaching the subject. In 1881–82 he added to his list a course on Vector Analysis, having apparently not given Least Squares for some years. The list continues with minor modifications through the catalog of 1885–86. For the year 1886–87 we find the first pretentious catalog, much larger than the preceding ones with better descriptions of the offerings. The list for Gibbs is (1) Vector Analysis, (2) Potential Theory, (3) Mathematical Theory of Electricity and Magnetism, (4) Electromagnetic Theory of Light, (5) The a priori Deduction of Thermodynamic Principles from the Theory of Probabilities, and it so continues through the year 1891–92, except for the addition of a course on the Computation of Orbits.

It is of more than passing interest that the classical thermodynamics represented by his own contributions has not appeared for the fifteen years since his paper was printed and that the first course announced by him in this field is apparently really his Statistical Mechanics on which nothing was printed until 1901. In the years 1892–94 he apparently offered a combination of classical thermodynamics with statistical mechanics and only from 1894–95 on came to divide the work into a course on his great memoir with a supplementary one on statistical mechanics. In the meantime he had added an option in advanced vector analysis and another in multiple algebra. Thus after the middle nineties he may be considered to have run the cycle: (1) Vector Analysis, (2) Advanced Vector Analysis, (3) Multiple Algebra, (4) Thermodynamics and Properties of Matter, (5) Statistical Mechanics, (6) Electromagnetic Theory of Light, (7) Potential Theory, and Theory of Electricity. Of these (1), (4), (6) were generally two hours per week throughout the year while the others were one hour per week. He seems to have taught about six hours per week, giving (1) yearly with (4) or (6) in alternate years, and adding on occasion one or two of the other four one-hour courses. During the three years 1899–

1902 I was fortunate enough to take all these subjects, except that the statistical mechanics (5) was not given separately, but was represented as some ten lectures at the conclusion of his thermodynamics (4).

Except for the Vector Analysis, I, in common with all Gibbs' students of my time, was ill prepared for his work. It was not infrequently the case that a student repeated the work to become more familiar with it, and it certainly was my intention to repeat most of the courses after my return from Paris. The instruction was not poor, but the concentration of thinking of the instructor was great. Once in a while Gibbs would get lost in a demonstration. He lectured without notes and what specific preparation he generally made I do not know. It was almost always some very simple affair on which he would go astray rather than something recondite. The year I took thermodynamics he could not make his Carnot engine run right. There was a tradition, perhaps unwarranted, that the Carnot engine was apt to trouble him. Sometimes he would unravel his difficulty before the end of the hour and it was then an especial treat to see his mind work; sometimes the end of the hour would come sooner and he would have to leave the matter over until the next time when he would appear with a sheet of paper containing the demonstration.

I do not believe that Gibbs kept much in the way of notes. I imagine that he wrote the closely reasoned and highly mathematical *Statistical Mechanics* out of his head (rather than from notes accumulated during previous years) between the time in the autumn of 1900 when he agreed to produce the book and the time in the summer of 1901 when he delivered the manuscript. The reason for this belief lies in the fewness and in the character of the papers he left when he died—there was practically no *Nachlass*. And yet he was known to be working on a program of publication. I know this because of the conversation I had with him in June 1902 when I was leaving for Paris for a year's study. It was by far the longest conversation I ever had with him, and of course the last. He said that he did not wish to determine my line of future interest but that he hoped I would consider taking some work in applied mathematics in Paris in addition to any I might take in pure mathematics. He ventured the opinion that one good use to which anybody might put a superior training in pure mathemat-

ics was to the study of the problems set us by nature. He remarked that in the thirty years of his professorship of mathematical physics he had had but a half-dozen students adequately prepared to follow his lectures. He did me the honor to include me in the list, though I myself never felt that my preparation in physics had been adequate. I asked why he had given exclusively such advanced courses, why he had not offered some more elementary work to prepare his students. He replied that he had not felt called upon to do so but that if I were willing he would be glad to have me look forward to giving upon my return a general introductory course on mathematical physics, and at any rate he would be happy if I would bear the possibility in mind while abroad. He then went on to say that if I should choose to occupy myself somewhat seriously with mathematical physics he had a considerable number of problems on which he thought I could make progress and that he would be glad to talk about them on my return. How much I have regretted that he did not talk of them at the time, but he gave no inkling of them.

Finally he proceeded to say something of his own plans for the future. He remarked that if he could depend on living to be as old as Methuselah he would continue to study for several hundred years yet, but that as he could not expect any such span of years he had decided to set about preparing some matters for publication. There were three lines of activity he desired to pursue: (1) The revision and extension of his work on thermodynamics, to which he said he had some additions to make covering more recently discovered experimental facts not yet adequately incorporated into the theory and other additions of theory apparently not yet exemplified in experiment. (2) A contribution to multiple algebra on which he said he had some ideas he thought worth while even though the subject appeared at the time not to be of much interest to mathematicians, most of whom were devoting their attention to analysis. (3) A revision of his method of computing orbits which should certainly be revised now that it had recently been printed verbatim by Buchholz in the third edition of Klinkerfues' Astronomy when certain important improvements were only too obvious. He asked what I should think he had best first undertake, but without waiting for reply answered that the astronomers were conservative and unlikely to be appreciative of improvements in his methods for

orbits, that the mathematicians were not impatient to learn of his ideas in multiple algebra and that on the whole he felt it was more important to set about the work in thermodynamics to which he had made no published contribution of significance for about twenty-five years.

Ten months later, in April 1903, Gibbs died. There were found among his papers some chapter headings and the first beginnings of text on the revision and extension of his thermodynamic work. It was clear that what he intended to accomplish he carried in his head and not on paper. We shall never know what he had in mind in any of the three lines of activity. He waited and studied too long. This situation is primarily that on which I base my opinion that he wrote the *Statistical Mechanics* out of his head in something like nine months in addition to his regular teaching. The task was serious. All through the winter and spring of 1900–1901 he worked not only by day, the light in his study in Sloane could be seen burning at night. The manuscript was finished in the summer at Intervale, N. H. After Gibbs died A. W. Phillips told me that it was this severe work that killed him. He said that they had gone together to the express office to dispatch the copy to Scribners, that up to that time Gibbs had been quite himself but that from the time they turned away from the office he slumped, the elasticity was gone from his gait, he was a worn out old man, and never fully came back.

This is a thrilling story but sad. However, it may not be true. I communicated it to my old friend Ralph Van Name, nephew of Willard Gibbs, who writes: "This may be true, but it was not apparent to his family," and later, "my comment on the incident of the delivery of the manuscript of the *Statistical Mechanics* was not made in a spirit of criticism, but merely as a statement of my recollection, and of that of my sister, whom I had consulted about it. Though both of us were in Europe at the time of Willard Gibbs' death, I did not leave New Haven until June 1902, and she not until March 1903. It is unquestionably true that my uncle worked to the limit of his strength in trying to get the volume finished on time, and that he did not get over the effects for a good while. But both of us have the impression that he seemed to be in practically normal health and spirits by the autumn of 1901. . . . My uncle's final illness

was a sudden and acute attack of a nature which has no obvious connection with his overwork two years before—it was an intestinal obstruction which the doctors were unable to relieve." I may say that all through the academic year 1901–2 Gibbs seemed to me to be in normal condition, and in his conversation of June 1902 of which I have given so long an account seemed to be looking cheerfully and healthily ahead with real pleasure in the prospect which he was outlining and with no discernible feeling that it might not be finished—indeed he spoke as one surely counting on being active on my return fifteen months later.

If I have gone at such length into this story I have done so chiefly because it so well illustrates stories which come with the best intention of truth from persons near to Gibbs, with just as high desires to tell the truth and nothing but the truth as I have on this occasion, but which none the less cannot be wholly credited, quite as I do not wholly credit as fact my own statements. There is the story that at home, where he lived all his life with his sister who had married his friend and classmate Addison Van Name, he always insisted on mixing the salad on the ground that he was a better authority than the others on the equilibrium of heterogeneous substances. A very pretty conceit, and one vouched for by a colleague much closer to Gibbs than I, but I daresay both the fact and the statement of reason for the fact would not be substantiated by the family. Another story refers to his letter to Nature in comment on and disproof of Lord Kelvin's proposed experiment to determine the velocity of longitudinal waves in the ether. It is said that when a colleague told him that he had just seen the letter in print Gibbs blushed and said that he could not believe the Editor of Nature would print it. That illustrates his modest and retiring disposition, which was a conspicuous trait, but seems hardly credible.

One often hears lament at Yale and elsewhere that Gibbs' colleagues did not capitalize his great discoveries in physical chemistry by developing the subject experimentally and intensively in New Haven from 1876 on. The comment often takes the turn of wondering how much greater role American science would have played in the growth of physical chemistry if Gibbs had accepted the offer to go to the Johns Hopkins University instead of remaining at Yale. How much difference would it

have made? Perhaps very little. What efforts Gibbs made to develop physical chemistry at Yale I do not know; perhaps none. That he knew his thermodynamic work was important and knew so when he printed it I have no doubt; but I have noted above that he appeared not to have lectured upon it in his cycle of courses for about 15 years after its completion, preferring for some reason to teach other subjects, and the subject matter of the memoir is not such as would be likely to diffuse around any university without exposition by the master unless by chance there were at hand some almost equally competent person who very much needed the work as a basis for his own, and knew that he needed it. Gibbs was not an advertiser for personal renown nor a propagandist for science; he was a scholar, scion of an old scholarly family, living before the days when research had become *r*é*s*earch. Probably he had faith that when the time was ripe for his thermodynamics, the doctrine would spread.

Another beautiful legend is that Gibbs was not appreciated in this country or at Yale during his life. It is probably true that his name was not well known to the ordinary Yale alumnus before the recent time when his photograph and some eulogy of him were widely circulated to the alumni during a drive for funds. But the efforts which were made to arrange for printing his long and costly paper in the Transactions of the Connecticut Academy in 1876–78 were a high testimonial to the faith of his local contemporaries in his work. He was elected to the National Academy at 40, the average age of election being 50, and only the year after the appearance of the second half of the thermodynamic memoir. In 1881 he received the Rumford Medal from the American Academy of Arts and Sciences which means that a group of his contemporaries in Boston appreciated promptly and highly his contributions in the field of heat. Of course he did not have the notice which Einstein receives today; he had no press agents and surely wanted none. There seems to be every evidence that he received the type of recognition to be expected.

Whether the establishment of the Gibbs Lectureship by the American Mathematical Society should be ranked as one of the honors to his memory or whether it belongs with the circularizing of his photograph to Yale alumni as an attempt to get some-

thing through exploiting his name I will not venture an opinion. It is well known that this Society through all of its life has been chiefly in control of those interested in research in pure mathematics. It is also well known that the group of American students who went to Germany to study mathematics in the late eighties and early nineties, at the very time when Klein was emphasizing the need in Germany of a greater attention to applied mathematics, came back to this country with a determination to promote only pure mathematics. This may have been wise at the time. American mathematicians had been too exclusively interested in the applications. We needed emphasis, perhaps temporarily over-emphasis, on pure theory and rigorous procedures, on analysis as it had developed in Europe. The lengths to which this emphasis was carried may be illustrated by my telling a story which happened not so long ago. When E. W. Brown, a past president of this Society, was at last naturalized as an American citizen and thus became eligible for election to our National Academy of Sciences, I asked one of the leaders in the section of mathematics of that Academy and also a leader in this Society whether the mathematicians would not nominate Brown to the Academy. He replied in a breezy vernacular "Not till Hell freezes over"—Brown was to him not a mathematician, but Brown is here in good company with Gibbs and G. W. Hill, H. A. Newton, Newcomb and others. Another story shows how non-mathematicians were impressed with this point of view. I once met at the Cosmos Club an eminent expert in international relations who asked why I happened to be in Washington. I replied "To attend the meeting of the National Academy of Sciences." "But," said he, "you are a mathematician not a scientist," and added, "oh, yes, I remember now that it is an academy of the sciences and of mathematics."

The mathematician has a dilemma, a choice. Insofar as he turns his attention to the abstract theory of his subject he is not a scientist dealing with observed fact but a philosopher playing with a priori hypotheses. It is only when he turns his attention to applied mathematics that he becomes a scientist. With the obvious need of specialization today, I would not limit the choice of the individual mathematician; he should be free to follow his bent or the exigencies of the institution he serves. A great national mathematical society, however, should not limit

its interests, it should cover the whole field of mathematics both pure and applied. In doing so it puts itself right on the pinnacle of intellectual effort. There is no problem requiring more brains, sounder judgment, better total adjustment internal and external, then that of uniting the logical and operational techniques of pure mathematics with the infinite variety of observable fact which Nature offers to our contemplation with a ringing challenge to our best abilities. It was in this field that Gibbs was supreme. He had studied with Weierstrass and was not unmindful of mathematical rigor; in the paper in which he pointed out that phenomenon of the convergence of Fourier series which has come to be known as the Gibbs phenomenon he showed his appreciation of mathematical precision as he did on other occasions. But fundamentally he was not interested in rigor for itself, he was inspired by the greater problem of the union between reflective analytical thought and the world of fact. He did not feel that one should not study pure mathematics; he was not one-sided or dogmatic in any of his views. What he said was that *one* of the uses of a good mathematical background was in the study of the problems set us by Nature. And if I had one special inference to draw from my contact with him to give you today it would be that the American Mathematical Society should follow for its own good his judgment on that matter. There are indeed indications that times are somewhat changing and that in the future the mathematics in which you as a Society are interested will be all mathematics, pure and applied. When that time comes there will be no possibility of raising the query as to whether the Gibbs Lectureship may be only lip service to a great name, for it will be evident that the thought of this Society is itself in no small measure a constant testimonial to that great thinker, J. Willard Gibbs.

You may be somewhat disappointed that I have no very striking personal reminiscences to recount, but what should you really expect in the way of impressions gained by an immature young fellow in the early twenties of a mature quiet scholar 40 years his senior? Gibbs was not a freak, he had no striking ways, he was a kindly dignified gentleman. I came to his courses in the days prior to tutorial systems when students were not expected to take the time of their teachers outside the classroom for personal contact and when teachers did not feel a moral urge

to guide their students otherwise than by instruction given in course. I am not sure but this was better for both student and teacher even if it has resulted in a less picturesque address today than one which some now twenty-year-old at the California Institute of Technology could give thirty years hence about Millikan.

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ON THE JOIN OF TWO COMPLEXES*

BY A. B. BROWN†

1. *Introduction.* Notations are as in Lefschetz's colloquium Publication.‡ Where not otherwise specified, cells and spheres are combinatorial.

In Chapter II of Lefschetz I the join of two complexes is defined, and it is proved that the join of two cells, or of a cell and a sphere, is a cell. We shall prove that if the given cells are normal, the join is likewise normal. In the later part of the paper we obtain formulas for the Betti numbers of the join of any two complexes.

2. *The Join of two Cells.*

THEOREM 1. *The join of the closures of two normal cells is the closure of a normal cell.*

PROOF. Since Lefschetz proves§ that the join is a cell, it remains only to prove that it is normal, that is, the join of a point and a sphere.

Let A and B denote the closed cells. Since they are normal, we may consider them to be located in a euclidean m-space; composed of simplexes; each having only one interior vertex, called C_a and C_b respectively; and such that (A, B) consists of A, B and the points on non-intersecting line segments, called *elements*, joining the points of A to the points of B.

In Lefschetz I, pp. 112–113, it is shown that $(A, F(B))$ and $(F(A), B)$, locus of the chain boundary of (A, B), constitute a sphere. We shall call them the *boundary* of (A, B), and the remaining points the *interior* of (A, B). Let C be an interior point of (C_a, C_b). We shall prove that (A, B) is the join of C with the boundary.

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* Presented to the Society, February 28, 1931.

† Part of the work on this paper was done while the author was a National Research Fellow, at Princeton University.

‡ S. Lefschetz, *Topology*, New York, 1930. (Lefschetz I.)

§ Lefschetz I, pp. 111–112.

Let P be any point on (A, B) not C itself. We shall prove: (1) interior points of (P, C) are interior points of (A, B); (2) if P is an interior point of (A, B), then (C, P) can be continued in a straight line beyond P in the interior of (A, B).

CASE I: P is an interior point of (P_a, P_b), where P_a and P_b are points of A and B distinct from C_a and C_b, respectively. Now C_a, C_b, P_a, P_b must determine a non-degenerate tetrahedron, since if they lay in any 2-plane we could find two non-identical elements having a common interior point. From the construction of a tetrahedron as the join of either pair of opposite edges, it follows that interior points of (C, P) are interior points of the tetrahedron, consequently are interior points of joins of pairs of points one on the interior of each of the segments (C_a, P_a) and (C_b, P_b). Since such points are interior points of (A, B), we conclude that (1) is proved.

In case P is an interior point of (A, B), then P_a and P_b must be interior points of A and B, respectively. Hence we can find points P_a' and P_b' on A and B, respectively, such that P_a and P_b are interior points of (C_a, P_a') and (C_b, P_b'), respectively. Then P will be an interior point of the tetrahedron determined by C_a, P_a', C_b, P_b'; and since (C, P) consists of points in the interior of that tetrahedron, we conclude that (2) is valid.

CASE II: All other cases. The proofs here are simpler, due to the fact that we have to deal with triangles at worst, instead of tetrahedrons as in the case just considered. Hence we omit the details.

From (1) we conclude that the interior points of the segments joining C to points of the boundary of (A, B) are themselves points of (A, B), and none of them is obtained more than once by the process. From (2) and the fact that (A, B) is in a bounded part of m-space, we infer that all the points of (A, B) are obtained in this way, that is, on joins of C with points of the boundary of (A, B). Since this boundary is a sphere, it follows that (A, B) is the closure of a normal cell; that is, Theorem 1 is proved.

3. *The Join of a Cell and a Sphere.*

THEOREM 2. *The join of a sphere and the closure of a normal cell is the closure of a normal cell.*

PROOF. Let S and E denote the sphere and the closed cell,

respectively. As above, we take them composed of simplexes in a euclidean m-space, such that E has only one interior vertex and (S, E) consists of the points on the line segments joining points of S with points of E. The locus of the chain boundary of (S, E) is $(S, F(E))$, a sphere,* which we shall call the *boundary* of (S, E). All other points of (S, E) will be called *interior* points.

Let C be the interior vertex of E, and P any point of (S, E) other than C itself. From this point the proof continues as in the case of Theorem 1. In Case I, P is an interior point of (P_s, P_e), where P_s is a point on S and P_e is a point on E distinct from C. In Case II, P is either on the join of C with a point of S, or on E. In both cases the proofs are simpler than the corresponding proofs for Theorem 1, involving triangles at worst. We shall give no further details.

4. *Betti Numbers of the Join of two Complexes.*

THEOREM 3. *Given complexes A and B, let A_i, B_i, P_i, J_i denote the ith Betti numbers of A, B, their product and their join, respectively. Then*†

$$J_i = P_{i-1} - B_{i-1} - A_{i-1} + \delta^0_{i-1}, \quad i \geqq 1;$$

$$J_0 = 1.$$

These formulas hold also for Betti numbers mod p, *p any prime greater than unity.*

PROOF. Let A and B be regularly subdivided, and images composed of simplexes taken in a euclidean space so that (A, B) is obtained by joining the points of A to the points of B by straight line segments, called elements. We introduce an additional coordinate, say z, keeping the other coordinates of the points of A and B fixed, and place A in the hyper plane $z=1$ and B in the plane $z=-1$. Let C denote the intersection of (A, B) with the plane $z=0$.

Let A_1 denote the part of (A, B) for which $z \geqq 0$, and B_1 the

* Lefschetz I, p. 112.

† $\delta_j^i = 1$ or 0 according as $i=j$ or $i \neq j$.

part for which $z \leqq 0$. Then we have, from Lemmas 1, 2, 4 and 6 in an earlier paper,* the following equalities:

$$R_i(A_1) = a^i + c_1{}^i + c_b{}^i; \tag{1}$$

$$R_i(B_1) = b^i + c_1{}^i + c_a{}^i; \tag{2}$$

$$R_i(C) = c_1{}^i + c_a{}^i + c_b{}^i + c_d{}^i; \tag{3}$$

$$R_i(A, B) = a^i + b^i + c_1{}^i + c_d{}^{i-1}. \tag{4}$$

Now a^i is the maximum number of i-cycles on A_1 that are independent† of the i-cycles on C. Any i-cycle on A_1 is homologous to one on A, since it can be deformed onto A along the elements. We can then deform the resulting i-cycle along the elements joining A to a single point of B, as far as the locus $z=0$, that is, up to C. Consequently every i-cycle on A_1 is homologous to one on C. Therefore $a^i=0$. By symmetry, $b^i=0$.

Any i-cycle on C can be deformed first onto A along the elements, then to any point of B along the elements joining A to that point of B. Since $c_1{}^i$ is the maximum number of i-cycles on C which are independent on D, we conclude that $c_1{}^i=0$, for $i \geqq 1$. Since (A, B) is connected, $c_1{}^0=1$.

The locus C is a homeomorph of the product of A and B. Hence $R_i(C)=P_i$, under our notation.‡ Since A_1 can be deformed onto A along the elements, it follows from Theorem 2§ of our paper cited above that $R_i(A_1)=R_i(A)$. By symmetry, $R_i(B_1)=R_i(B)$.

By substituting the values just obtained in (1), (2) and (3), and then substituting from (1), (2) and (3) in (4), we obtain the first relation of Theorem 3. The second is a consequence of the fact that (A, B) is connected.

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* A. B. Brown, *Relations between the critical points of a real analytic function of n independent variables*, American Journal of Mathematics, vol. 52 (1930), pp. 251–270. The lemmas in question, stated for Betti numbers absolute and mod 2, hold also for Betti numbers mod p, where p is any prime greater than unity; as do the proofs. The meanings of some of the symbols are explained below.

† Independence refers to homologies.

‡ See Lefschetz I, Chap. 5. $P_i=\sum_{r+s=i}A_rB_s$.

§ This theorem holds for Betti numbers mod m, m any integer greater than unity. The proof is easily modified to cover this case.

ON THE NUMBER OF APPARENT DOUBLE POINTS OF r-SPACE CURVES

BY B. C. WONG

Consider a curve C^N of order N in r-space. The number, h, of $(r-2)$-spaces passing through a given $(r-3)$-space and meeting C^N twice is finite. If C^N is projected on to a 3-space, then h is the number of apparent double points on the projection. To avoid circumlocution, we shall use the phrase *the apparent double points of* C^N instead of *the apparent double points of the 3-space projection of* C^N.

When the curve C^N is the intersection of $r-1$ hypersurfaces of order $n_1, n_2, \cdots, n_{r-1}$, the number of its apparent double points is known and is given by the formula*

$$(1) \qquad h = \tfrac{1}{2}n_1n_2\cdots n_{r-1}(n_1n_2\cdots n_{r-1} - \sum n_i + r - 2).$$

But suppose C^N is not the intersection of $r-1$ hypersurfaces but the intersection of $q<r-1$ varieties $V^{n_1}_{r_1}, V^{n_2}_{r_2}, \cdots, V^{n_q}_{r_q}$ of orders $n_1, n_2, \cdots, n_q$ and of dimensions (which may be different) $r_1, r_2, \cdots, r_q$ where

$$(2) \qquad r_1 + r_2 + \cdots + r_q = r(q-1) + 1.$$

What is the formula for h for such a curve? It is our purpose in this paper to derive this formula.

As a first step in the derivation, let $q=2$. Then C^N or $C^{n_1n_2}$ is the intersection of two varieties $V^{n_1}_{r_1}$, $V^{n_2}_{r_2}$, where $r_1+r_2=r+1$. Let h_i be the number of apparent double points on the curve C^{n_i} in which an S_{r_i} meets $V^{n_i}_{r_i}$. Decompose one of the given varieties, say $V^{n_1}_{r_1}$, into n_1 r_1-spaces having severally $\frac{1}{2}n_1(n_1-1)-h_1$ (r_1-1)-spaces in common. The curve C^{n_1} in which an S_{r_2} meets the decomposed $V^{n_1}_{r_1}$ is, then, composed of n_1 lines forming a skew n_1-sided polygon with $\frac{1}{2}n_1(n_1-1)-h_1$ vertices. Now the curve $C^{n_1n_2}$ in which $V^{n_2}_{r_2}$ meets the decomposed $V^{n_1}_{r_1}$ is composed of n_1

* Veronese, *Behandlung der projectivischen Verhältnisse der Räume von verschiedenen Dimensionen durch das Princip des Projicirens und Schneidens*, Mathematische Annalen, vol. 19 (1882), pp. 161–234. The formula above is given on p. 205.

curves all of order n_2. If any two of these n_1 curves intersect, they must intersect in n_2 points lying in one of the $\frac{1}{2}n_1(n_1-1)-h_1$ $(r-1)$-spaces mentioned above. Each of these (r_1-1)-spaces contains n_2 such points. Hence, the total number of points in which the n_1 curves actually intersect severally is seen to be $n_2[n_1(n_1-1)/2-h_1]$. The total number of intersections, both actual and apparent, of the n_1 curves two by two is $\frac{1}{2}n_1n_2^2(n_1-1)$. Now each of the n_1 curves has h_2 apparent double points. Therefore, we conclude that the number h of apparent double points on the curve $C^{n_1n_2}$, proper or improper, is equal to the sum of the number of apparent intersections of the component curves of the degenerate $C^{n_1n_2}$ and the total number of the apparent double points on the component curves, that is,

$$(3)\quad h = \tfrac{1}{2}n_1n_2^2(n_1-1) - n_2[n_1(n_1-1)/2 - h_1] + n_1h_2$$
$$= \tfrac{1}{2}n_1n_2(n_1n_2 - n_1 - n_2 + 1) + n_2h_1 + n_1h_2.$$

Suppose we have a curve $C^{n_1n_2n_3}$ which is the intersection of three varieties $V^{n_1}_{r_1}$, $V^{n_2}_{r_2}$, $V^{n_3}_{r_3}$ in S_{r_4}, where $r_1+r_2+r_3=2r+1$. Let h_i be the number of apparent double points on the curve C^{ni} in which an S_{r-r_i+1} meets $V^{ni}_{r_i}$. To find the number h of apparent double points on $C^{n_1n_2n_3}$, we may reason as above or we may proceed as follows.

The curve $C^{n_1n_2n_3}$ may be considered as the intersection of $V^{n_3}_{r_3}$ and the variety $V^{n_1n_2}_{r_1+r_2-r}$, the latter being the intersection of $V^{n_1}_{r_1}$ and $V^{n_2}_{r_2}$. Let h_{12} be the number of apparent double points on the curve $C^{n_1n_2}$ in which an S_{r_3} meets $V^{n_1n_2}_{r_1+r_2-r}$ and its value is given by (3). Applying formula (3), we find, replacing n_1, n_2, h_1, h_2 by n_1n_2, n_3, h_{12}, h_3 respectively,

$$h = \tfrac{1}{2}n_1n_2n_3(n_1n_2n_3 - n_1n_2 - n_3 + 1) + n_3h_{12} + n_1n_2h_3.$$

Writing for h_{12} its value from (3) in the above, we obtain

$$(4)\quad h = \tfrac{1}{2}n_1n_2n_3(n_1n_2n_3 - n_1 - n_2 - n_3 + 2)$$
$$+ n_2n_3h_1 + n_3n_1h_2 + n_1n_2h_3$$

as the number of apparent double points on $C^{n_1n_2n_3}$.

Now let $q=4$. Then C^N, where $N=n_1n_2n_3n_4$, is the intersection of four varieties $V^{n_1}_{r_1}$, $V^{n_2}_{r_2}$, $V^{n_3}_{r_3}$, $V^{n_4}_{r_4}$, where $r_1+r_2+r_3+r_4=3r+1$. We may regard C^N as the intersection of $V^{n_4}_{r_4}$ and the variety $V^{n_1n_2n_3}_{r_1+r_2+r_3-2r}$, the latter being the intersection of $V^{n_1}_{r_1}$, $V^{n_2}_{r_2}$, $V^{n_3}_{r_3}$, and

apply (3) and (4), or we may regard it as the intersection of a $V^{n_1n_2}_{r_1+r_2-r}$ and a $V^{n_3n_4}_{r_3+r_4-r}$, the former being the intersection of $V^{n_1}_{r_1}$, $V^{n_2}_{r_2}$ and the latter that of $V^{n_3}_{r_3}$, $V^{n_4}_{r_4}$, and then apply (3) alone. Adopting the latter view, we have, replacing n_1, n_2, h_1, h_2 by n_1n_2, n_3n_4, h_{12}, h_{34} respectively in (3),

$$h = \tfrac{1}{2}n_1n_2n_3n_4(n_1n_2n_3n_4 - n_1n_2 - n_3n_4 + 1) + n_1n_2h_{34} + n_3n_4h_{12},$$

where h_{12} and h_{34} are the respective numbers of apparent double points on the curves $C^{n_1n_2}$, $C^{n_3n_4}$ in which an $S_{r_3+r_4-r}$ and an $S_{r_1+r_2-r}$ meet the varieties $V^{n_1n_2}_{r_1+r_2-r}$ and $V^{n_3n_4}_{r_3+r_4-r}$ respectively. Now h_{12} is given by (3) and h_{34} is also given by (3) if n_1, n_2, h_1, h_2 are replaced by n_3, n_4, h_3, h_4. Making these substitutions in the above, we have

$$(5)\quad h = \tfrac{1}{2}n_1n_2n_3n_4(n_1n_2n_3n_4 - n_1 - n_2 - n_3 - n_4 + 3) + n_2n_3n_4h_1 + n_3n_4n_1h_2 + n_4n_1n_2h_3 + n_1n_2n_3h_4.$$

Without going through any further details we give at once the following formula, which can be easily verified, for the number of apparent double points on a curve C^N, where $N = n_1n_2 \cdots n_q$:

$$(6)\quad h = \tfrac{1}{2}n_1n_2 \cdots n_q(n_1n_2 \cdots n_q - \sum n_i + q - 1) + n_1n_2 \cdots n_q \sum^{q} h_i/n_i.$$

If $q = r-1$, we have, from (2), $r_1 = r_2 = \cdots = r_{r-1} = r-1$. Then the curve C^N is the intersection of $r-1$ hypersurfaces. In this case, $h_1 = h_2 = \cdots = h_{r-1} = 0$ as a plane section of a hypersurface cannot have apparent double points. Then (6) is reduced to (1).

As an illustration, let C^9 be the intersection of a V^3_3 and a V'^3_3 in S_5. Since an S_3 in S_5 meets V^3_3 and V'^3_3 each in a twisted cubic curve, we have $h_1 = h_2 = 1$. We may use (3) or we may use (6) for $q = 2$. Putting $n_1 = n_2 = 3$, we have $h = 24$ as the number of apparent double points on the curve C^9.

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THE EQUIVALENCE OF TRIPLES OF BILINEAR FORMS*

BY O. E. BROWN

1. *Introduction.* If, in a bilinear form with matrix† A, we subject the x's to a linear transformation with the matrix C and the y's to a linear transformation with the matrix B, we obtain a new bilinear form with the matrix $C'AB$, where C' is the transpose of C. Then two bilinear forms are equivalent if and only if their matrices are equivalent. In this paper we direct our attention entirely to the equivalence of matrices.

It is well known that two $m \times m$ matrices are equivalent if and only if they have the same rank. From classical theory we have also the theorem that two pairs of $m \times m$ matrices are equivalent in the sense of Weierstrass if and only if they have the same invariant factors. We are interested in the extension of the problem to the equivalence of triples of matrices.

2. *The General Problem. Given two triples of $m \times m$ matrices*

$$(a_1, a_2, a_3), \quad (\alpha_1, \alpha_2, \alpha_3),$$

to find necessary and sufficient conditions that there exist two non-singular matrices P and Q such that

$$Pa_1Q = \alpha_1, \quad Pa_2Q = \alpha_2, \quad Pa_3Q = \alpha_3.$$

The problem was studied by Sherwood‡ who found certain invariants for triples of orders two and three, which, as he says, are not sufficient to characterize all such triples.

It can be easily shown that in the general problem we are at

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* Presented to the Society, March 30, 1929.

† It is understood throughout this paper that the elements of all matrices are in the field of all complex numbers.

‡ G. E. F. Sherwood, *Equivalence of triples of bilinear forms*, University of Chicago Doctor's thesis (1922). Mention is made by Sherwood of two papers by S. Kantor, *Theorie der Äquivalenz linear ∞^λ Schaaren bilinear Formen*, Sitzungsberichte der Akademie, München, vol. 27 (1897), and *Theorie der Elementartheiler höherer Stufen*, Monatshefte für Mathematik und Physik, vol. 11 (1900).

liberty to replace each matrix in one of the triples by a linear combination of the three matrices provided the three linear combinations taken are linearly independent and provided we make the same change in the other triple. If the net $la_1+ma_2+na_3$ contains a non-singular matrix it may be reduced to the identity by a preliminary transformation upon the triple. Let us, then, impose the condition that the *net contain a non-singular matrix.*

3. *The Restricted Problem. Given two triples of $m\times m$ matrices*

$$(I, a, b), \quad (I, \alpha, \beta),$$

to find necessary and sufficient conditions that there exist a non-singular matrix P such that

$$PaP^{-1} = \alpha, \quad PbP^{-1} = \beta.$$

We shall say that M is a *transformation* changing a to α if M is a non-singular matrix such that $MaM^{-1}=\alpha$. Furthermore, if such an M exists, a and α will be called *similar*. In the sequel we shall have use for the following lemma.

LEMMA. *If a, μ, and α are three mutually similar matrices and if P, Π and T are transformations changing a to μ, α to μ and a to α, respectively, then we can write*

$$T = \Pi^{-1}GP,$$

where G is a transformation leaving μ unaltered.

The hypotheses of the lemma take the form $PaP^{-1}=\mu$, $\Pi\alpha\Pi^{-1}=\mu$, $TaT^{-1}=\alpha$. It is obviously true that

$$T = \Pi^{-1}\Pi TP^{-1}P$$

and by use of the last of the three given relations we obtain

$$\Pi^{-1}\Pi TP^{-1}PaP^{-1}PT^{-1}\Pi^{-1}\Pi = \alpha.$$

If we multiply on the left by Π and on the right by Π^{-1} and use the other two given relations we obtain

$$\Pi TP^{-1}\mu PT^{-1}\Pi^{-1} = \mu.$$

Then ΠTP^{-1} leaves μ unaltered and may be identified with G, which establishes the lemma.

4. *Necessary Conditions.* Let us suppose that the two triples (I, a, b), (I, α, β) are equivalent and that M is the transformation such that

$$MaM^{-1} = \alpha, \quad MbM^{-1} = \beta.$$

Then a and α must be similar to a canonical form μ while b and β must be similar to a canonical form ν, μ and ν being determined by the invariant factors of the characteristic matrices $(a-\lambda I)$ and $(b-\lambda I)$. Now let P, L, Π and Λ be transformations such that

$$LaL^{-1} = \mu, \quad PbP^{-1} = \nu, \quad \Lambda\alpha\Lambda^{-1} = \mu, \quad \Pi\beta\Pi^{-1} = \nu.$$

By the preceding lemma we have

$$M = \Lambda^{-1}G^{-1}L = \Pi^{-1}H^{-1}P,$$

where G leaves μ unaltered while H leaves ν unaltered. If we multiply on the right by L^{-1} and on the left by $H\Pi$ we have

$$PL^{-1} = H\Pi\Lambda^{-1}G^{-1}.$$

5. *Sufficiency Conditions.* By reversing the steps in the above argument we see that if this equation is satisfied the triples are equivalent.

6. *Notation.* Let us call PL^{-1} and $\Pi\Lambda^{-1}$ *transpositions* of the first and second triples, respectively. The transformation $H\Pi$ changes β to ν as required of Π, while $G\Lambda$ changes α to μ as required of Λ. Hence $H\Pi\Lambda^{-1}G^{-1}$ is also a transposition of the second triple. If, under the above circumstances, there exists a transformation G leaving μ unaltered and a transformation H leaving ν unaltered such that $H\Pi\Lambda^{-1}G^{-1} = PL^{-1}$, let us say that the transpositions PL^{-1} and $\Pi\Lambda^{-1}$ are *cognate.* In this nomenclature we may state our theorem.

THEOREM. *Two triples of $m \times m$ matrices (I, a, b) and (I, α, β) with a similar to α and b similar to β are equivalent if and only if their tranpositions are cognate.*

One may note that if μ and ν are construed as in the classical canonical form, and if the characteristic equations $a-\lambda I = 0$ and $b-\lambda I = 0$ have no multiple or zero roots, then μ and ν are diagonal matrices and so, therefore, G and H are also. In this case as in many others the relation *cognate* is easily recognized.

CASE SCHOOL OF APPLIED SCIENCE

GENERALIZATION OF A THEOREM OF KRONECKER

BY B. L. VAN DER WAERDEN

EXTRACT FROM A LETTER TO J. F. RITT

Your theorem on algebraic dependence* is, in the very special case in which all $\alpha_i, \beta_i, \gamma_i$ are powers of y, contained in a theorem of Kronecker.† Emmy Noether communicated to me some years ago a proof of Kronecker's theorem which can be extended as below, to your more general case. This proof is simpler than yours, and gives more information. The products $b_i c_j$ are not only algebraic, but *integral* algebraic.

HYPOTHESIS. *Let* $\beta_1, \cdots, \beta_r$; $\gamma_1, \cdots, \gamma_s$ *be two systems of linearly independent analytic functions of* y. *Let* $b_1, \cdots, b_r$; $c_1, \cdots, c_s$ *be indeterminates. Let*

$$(1)\quad (b_1\beta_1 + \cdots + b_r\beta_r)(c_1\gamma_1 + \cdots + c_s\gamma_s) = (a_1\alpha_1 + \cdots + a_n\alpha_n),$$

where the α*'s are a linearly independent‡ set of products* $\beta_i\gamma_j$ *in terms of which all such products are expressible, and where the* a*'s are linear combinations of the products* $b_i c_j$.

CONCLUSION. *Every* $b_i c_j$ *satisfies an equation of the form*

$$(2)\quad Z^t + A_1 Z^{t-1} + \cdots + A_t = 0,$$

with every A_k *a homogeneous form of the* k*th degree in* $a_1, \cdots, a_n$, *with constant coefficients.*

PROOF. If the expressions $a_1, \cdots, a_n$ are all zero for special values $b_1', \cdots, b_r'$; $c_1', \cdots, c_s'$ of the indeterminates b_i, c_j, it follows from (1) that

$$(3)\quad (b_1'\beta_1 + \cdots + b_r'\beta_r)(c_1'\gamma_1 + \cdots + c_s'\gamma_s) = 0.$$

But if a product of analytic functions vanishes identically, one of the factors vanishes identically. If the first factor in (3) is zero, every b_i' is zero; if the second factor vanishes, then every c_i' does. In any case every product $b_i' c_j'$ vanishes. This means

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* *On a certain ring of functions of two variables*, Transactions of this Society, vol. 32 (1930), p. 155.

† Berliner Sitzungsberichte, vol. 37 (1883), p. 957. (NOTE by J. F. Ritt: This relationship was known to me, but I was not in possession of the elegant methods of proof which Professor van der Waerden uses below.)

‡ With respect to all constants.

that every zero* of the ideal $(a_1, \cdots, a_n)$ is also a zero of the ideal

$$\theta = (\cdots, b_i c_j, \cdots) = (b_1, \cdots, b_r)(c_1, \cdots, c_s).$$

It follows from a well known theorem of Hilbert† that, for some positive integer t; $\theta^t \equiv 0$, $(a_1, \cdots, a_n)$, or, if one designates the products $b_i c_j$, in any order, by $d_1, \cdots, d_{rs}$,

$$d_{i_1} d_{i_2} \cdots d_{i_t} = \sum e_{i_1 \cdots i_t k} a_k. \tag{4}$$

As the a's are linear combinations of the d's, the coefficients e in (4) may be taken‡ as homogeneous forms of degree $t-1$ in the d's. If the power products of the d's of degree $t-1$, written in any order, are designated by $g_1, \cdots, g_k$, then (4) may be written in the form $d_i g_j = \sum g_l a'_{jl}$, where the a'_{jl} are linear combinations of the a's. Elimination of the g's gives

$$\begin{vmatrix} d_i - a_{i11} & -a_{i12} & \cdots \\ -a'_{i21} & d_i - a'_{i22} & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{vmatrix} = 0.$$

This is an algebraic equation for d_i of the form (2). The theorem is then proved.

If, now, the indeterminates b_i, c_j in (1) are replaced by other quantities, for instance functions of x, not necessarily analytic, the a's in the second member may become linearly independent. If they are all expressed in terms of the linearly dependent ones among them, the second member appears in "reduced form" (*On a certain ring*, etc., p. 157). Equation (2) holds identically and hence preserves its form when the b's and c's are replaced by other quantities, even if the a's are expressed in terms of the linearly independent ones among them. This proves your Theorem 1 (loc. cit., p. 156) with the additional information that the $b_i c_j$ are *integral* algebraic in the a's.

GRONINGEN, HOLLAND

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* A set of values of the indeterminates for which every polynomial in the ideal vanishes.

† See Macaulay, *Modular Systems*, p. 46. (J. F. R.)

‡ The equation thus obtained may be written in the form of the Dedekind-Mertens "modulus-equation" $\theta^t = \theta^{t-1}\alpha$, $\alpha = (a_1, \cdots, a_n)$, which occurs in Dedekind's proof of Kronecker's theorem.

ON THE CYCLIC CONNECTIVITY THEOREM*

BY G. T. WHYBURN

1. *Introduction.* We shall call the following theorem the *cyclic connectivity theorem.*

Every two points of a locally connected continuum having no cut point lie together on a simple closed curve in that continuum.

The demonstration for this theorem originally given by the present author† for the case of plane continua and in particular the demonstration given later by Ayres‡ for the theorem in general space are undeniably quite complicated. Indeed, the complexity of the proof of this theorem constituted a strong incentive to the author to seek and find§ a new treatment of the cyclic element theory which not only avoids using this theorem as principal point of departure as does the original one‖ but also has validity in all connected, locally connected, metric, and separable spaces, and thus in spaces in which the proposition in question obviously does not hold. The same complexity was the prevailing influence motivating a development by Kuratowski and the author¶ of most of the cyclic element theory for compact locally connected continua in a simple and direct way independent of the cyclic connectivity theorem, based on a definition of *cyclic element* suggested by R. L. Moore.**

Thus it is seen that although this proposition has been almost successfully avoided in so far as the cyclic element theory is

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* Presented to the Society, February 28, 1931.

† See Proceedings of the National Academy of Sciences, vol. 13 (1927), pp. 31–38.

‡ W. L. Ayres, American Journal of Mathematics, vol. 51 (1929), pp. 577–594.

§ See Transactions of this Society, vol. 32 (1930), pp. 926–943.

‖ See American Journal of Mathematics, vol. 50 (1928), pp. 167–194.

¶ C. Kuratowski et G. T. Whyburn, *Sur les éléments cycliques et leurs applications*, Fundamenta Mathematicae, vol. 16 (1930), pp. 305–331. The authors of this article describe the proof of the cyclic connectivity theorem as being "fort compliquée."

** R. L. Moore, Monatshefte für Mathematik und Physik, vol. 36 (1929), pp. 81–88.

concerned, it has not been proved in the direct and simple manner which is characteristic of demonstrations for the majority of theorems concerning cyclic elements. The present paper offers as its principal contribution just such a demonstration for this theorem, based on a small amount of the cyclic element theory which, for the sake of completeness, is appended at the end of the paper, together with a few fundamental and long established properties of locally connected continua. The cyclic connectivity theorem thus finds its proper place in the subject as an important *consequence* of the cyclic-element decomposition of locally connected continua and an important complement to the cyclic element theory.

2. *The Proof.* Let M designate any locally connected, locally compact, separable and metric continuum, which we shall consider as a space, and let C designate any such space which has no cut point.

LEMMA 1. *If A and B are non-degenerate,* closed and mutually exclusive subsets of C, there exist two mutually exclusive arcs in C joining A and B.*

There exists an arc ab in C where $ab \cdot A = a$, $ab \cdot B = b$; and if p is a point of $A - a$, clearly there exist points x which can be joined in C to p by an arc px containing no point of ab. Thus there exist points x such that mutually exclusive arcs ab and px exist in C so that

$$(1) \qquad ab \cdot A \supset a, \quad ab \cdot B \supset b, \text{ and } px \cdot A \supset p.$$

Let S denote the set of all such points x. I shall show that $S = C$. Suppose this is not so. Then since C is connected and clearly S is open in C, it follows that at least one point y of $C - S$ is a limit point of S. There exists an arc $a'b'$ in $C - y$ such that $a'b' \cdot A = a'$ and $a'b' \cdot B = b'$. Let R be a region (= connected open subset of C) containing y but having no point or boundary point in $a'b' + A$. Then R contains a point x of S, and there exist arcs ab and px satisfying (1). Since y does not belong to S, it follows at once that $ab \cdot R \neq 0$. The arcs ab and px contain arcs aw and pr respectively such that $aw \cdot \overline{R} = w$ and $pr \cdot \overline{R} = r$. Let $H = A + aw + pr$.

* A point set is degenerate or non-degenerate according as it does or does not reduce to a single point. This terminology is due to R. L. Moore.

Then $a'b'$ contains an arc uv such that $uv \cdot H = u$ and $uv \cdot B = v$. Let T denote one of the sets aw and pr which does not contain u and let Z denote the other one of these sets. Let Q be a region containing the point $T \cdot \overline{R}$ and containing no point of $Z + uv$. Then $Z + uv$ contains an arc mn and $T + Q + R$ contains an arc qy such that $mn \cdot A \supset m$, mn, $B \supset m$, $qy \cdot A \supset q$, and $mn \cdot qy = 0$. But this is impossible since y does not belong to S. Therefore $S = C$. Accordingly S contains a point x of B, and thus there exist two mutually exclusive arcs ab and px joining A and B.

LEMMA 2. *Every point x of C is an interior point of some arc axb in C.*

This is obvious if x is a cut point of some region R in C; for then it is only necessary to take a and b in different components of $R - x$ and any arc ab in R will contain x. Thus we may suppose that x is a cut point of no region in C. Now let a and b be any two distinct points of $C - x$. Let R_1 be a region containing x of diameter < 1 so that $\overline{R}_1 \cdot (a + b) = 0$. There exists a locally connected subcontinuum E_1 of C of diameter < 1 which contains $\overline{R}_1$ but does not contain either a or b. Since x is not a cut point of R_1, it cannot be a cut point of E_1; and since it is not an end point of E_1 it therefore (see appendix below) lies in some non-degenerate cyclic element C_1 of E_1. By Lemma 1 there exist in C two mutually exclusive arcs aa_1 and bb_1 where $aa_1 \cdot C_1 = a_1$ and $bb_1 \cdot C_1 = b_1$. Obviously, we may suppose $a_1 \neq x \neq b_1$. Let R_2 be a region in C_1 containing x of diameter less than $1/2$, such that $\overline{R}_2 \cdot (a_1 + b_1) = 0$. There exists a locally connected subcontinuum E_2 of C_1 of diameter less than $1/2$ which contains $\overline{R}_2$ but does not contain either a_1 or b_1. Again we may suppose that x cuts no region in C_1; and it follows just as in the case of E_1 that x lies in some non-degenerate cyclic element C_2 of E_2. By Lemma 1 there exist in C_1 two mutually exclusive arcs a_1a_2 and b_1b_2 such that $a_1a_2 \cdot C_2 = a_2$ and $b_1b_2 \cdot C_2 = b_2$. Let R_3 be a region in C_2 of diameter less than $1/3$, and so on. If we continue this process indefinitely, it is clear that the point set

$$x + aa_1 + a_1a_2 + a_2a_3 + \cdots + bb_1 + b_1b_2 + \cdots$$

so obtained is an arc axb in C.

THEOREM. *C is cyclicly connected.*

Let a and b be any two points of C. Now every point x of C

lies on some simple closed curve in C; for by Lemma 2 there exists an arc pxq in C and there exists an arc pyq in $C-x$, and clearly $pxq+pyq$ contains a simple closed curve containing x. Thus there exist simple closed curves C_a and C_b in C containing a and b, respectively. Now if $C_a \cdot C_b = 0$, then by Lemma 1 there exist mutually exclusive arcs mn and uv in C where $mn \cdot C_a = m$, $mn \cdot C_b = n$, $uv \cdot C_a = u$, and $uv \cdot C_b = v$; and in this case clearly the set mn+arc nbv of C_b+uv+arc uam of C_a is a simple closed curve in C containing $a+b$. If $C_a \cdot C_b = p$, a single point, then $C-p$ contains an arc uv so that $uv \cdot C_a = u$ and $uv \cdot C_b = v$; and in this case uv+arc vbp of C_b+arc pau of C_a is a simple closed curve in C containing $a+b$. Finally, if $C_a \cdot C_b$ contains more than one point, then C_b contains an arc pbq, where $pbq \cdot C_a = p+q$; and in this case pbq+arc paq of C_a is a simple closed curve in C containing $a+b$. Thus the cyclic connectivity theorem is established.

3. *Appendix.* For the sake of completeness, proofs will now be given for that part of the cyclic element theory which has been used in the above demonstration of the cyclic connectivity theorem.

DEFINITION. A cyclic element of our space M is either a cut point of the space or a set M_p, where p is a non-cut point and M_p is the set of all points which are not separated from p by any single point. (See references in §1.)

(1) *$M_p = p$ only when p is an end point.*

For suppose $M_p = p$, and let ϵ be any positive number. Since p is not a cut point of M, there exists a δ, $0 < \delta < \epsilon$, such that $M - V_\epsilon(p)$ is a subset of some single component N of $M - V_\delta(p)$, where $V_\epsilon(p)$ denotes the set of all points whose distance from p is $< \epsilon$, and similarly for δ. Let pq be an arc such that $pq \cdot N = q$. There exists a point x which separates p and q, because $M_p = p$. Clearly $x \subset pq$. Thus $x \cdot N = 0$ and, as N is connected, x separates p and N. Hence x ϵ-separates p, and therefore p is an end point.

(2) *Every M_p has the property that each component N of $M - M_p$ has just one limit point in M_p.*

For if M_p contains two limit points of N, then since clearly M_p is closed, it follows that there exists an arc ab such that $ab \cdot M_p = a+b$. Hence if q is a point of $ab-(a+b)$, some point x separates p and q. But then x necessarily separates q and the set $M_p - x$, which is impossible since we have the subarcs qa and qb of ab joining q and the set M_p.

(3) *If the set Z is connected, so also is $Z \cdot M_p$ (when non-vacuous).*

If, on the contrary, $Z \cdot M_p = H_1 + H_2$, where H_1 and H_2 are mutually separated, then let Z be divided into two sets Z_1 and Z_2 in such a way that these sets contain H_1 and H_2, respectively, and any other point x of Z belongs to Z_1 or to Z_2 according as the boundary point of the component of $M - M_p$ containing x belongs to H_1 or to H_2. Obviously $Z_1 \cdot Z_2 = 0$. And if a point x of one of these sets, say of Z_2, is a limit point of the other, Z_1, then since, by virtue of

the local connectivity of M, any component of $M-M_p$ containing points of Z_2 is a neighborhood of any one of its points and contains no point of Z_1, it follows that x belongs to H_2. But then if V is a region containing x but containing no point of H_1, V contains a point y of $Z_1 \cdot (M-M_p)$, and clearly this is impossible because the boundary point of the component of $M-M_p$ containing y belongs to H_1 and hence not to V.

(4) *Every M_p is closed, connected, and locally connected and has no cut point.*

Taking $Z=M$, we have $Z \cdot M_p = M_p$; and thus by (3) it follows that M_p is connected. Similarly, since each pair of sufficiently near points of M_p lie together in a connected subset of M of arbitrarily small diameter and the product of this connected set by M_p is connected, it follows that M_p is locally connected. Obviously M_p is closed. Finally, if some point x cuts M_p, let a and b be points lying in different components of M_p-x. Now a and b must lie together in some component N of $M-x$, for otherwise x would separate either a or b from p in M. But by (3), $N \cdot M_p$ is connected, which is absurd, because N does not contain x. Therefore M_p has no cut point.

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IDEAL MULTIPLICATION IN A LINEAR ALGEBRA*

BY GRACE SHOVER AND C. C. MACDUFFEE

1. *Introduction.* A recent approach† to the theory of ideals through matrices with rational integral elements proved to be successful for domains of integrity of linear associative algebras over the rational field as well as for domains of algebraic fields. The great weakness of this method was the lack of an adequate treatment of ideal multiplication.

In the present paper it is shown that ideal multiplication can be treated in a satisfactory manner by means of the matric theory. The way now seems to open for considerable development of the theory of ideals in rational algebras.

It is particularly to be noted that the classical theory of ideals in an algebraic field is a special case of the present theory. By way of illustration it is shown that the method of this paper can be applied to find by a straightforward, rational process the canonical form of the product of two ideals.

2. *Definitions and References.* Let $\mathfrak{A}$ be a rational semi-simple algebra of order n, and $\mathfrak{S}$ a set of integral elements of order n in $\mathfrak{A}$.‡ We suppose that the basal numbers $e_1, e_2, \cdots, e_n$ of $\mathfrak{A}$ form a basis for $\mathfrak{S}$, e_1 being the principal unit.

If the constants of multiplication are c_{ijk}, that is, if

$$e_i e_j = \sum_k c_{ijk} e_k \qquad (i, j = 1, 2, \cdots, n),$$

then the matrices

$$R_i = (c_{isr}), \quad S_i = (c_{ris})$$

are called the *first* and *second matrices*§ of e_i. If $\xi = \sum x_i e_i$ is the general number, then

$$R(\xi) = \sum x_i R_i, \quad S(\xi) = \sum x_i S_i$$

form sets of matrices each of which gives a matric representation of the algebra.

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* Presented to the Society, December 30, 1930.

† C. C. MacDuffee, Transactions of this Society, vol. 31 (1929), pp. 71–90.

‡ Dickson, *Algebren und ihre Zahlentheorie*, 1927, p. 155.

§ MacDuffee, this Bulletin, vol. 35 (1929), p. 344.

A left (right) ideal $\mathfrak{K}$ is a set of numbers of $\mathfrak{S}$ which is closed under addition and subtraction, and under multiplication on the left (right) by the numbers of $\mathfrak{S}$. If not every number of $\mathfrak{K}$ is of norm* 0, $\mathfrak{K}$ is non-singular. We shall speak only of left ideals, as the theory of right ideals is obviously parallel.

Let $\mathfrak{K}_\alpha$ be an ideal with basis $(\alpha_1, \alpha_2, \cdots, \alpha_n)$. Then

$$\alpha_i = \sum_{j=1}^{n} g_{ij} e_j,$$

where the g_{ij} are rational integers. We shall call $G_\alpha = (g_{rs})$ an *ideal matrix* corresponding* to the ideal $\mathfrak{K}_\alpha$. If $|g_{rs}| \neq 0$, then $\mathfrak{K}_\alpha$ is non-singular. If G_α corresponds to the non-singular ideal $\mathfrak{K}_\alpha$, then the totality of ideal matrices corresponding to $\mathfrak{K}_\alpha$ is given by AG_α, where A is a matrix with rational integral elements (in brief, an integral matrix) of determinant ± 1.

If $\mathfrak{K}_\alpha$ and $\mathfrak{K}_\beta$ are two ideals, we define the ideal product $\mathfrak{K}_\alpha \mathfrak{K}_\beta$ to be the set of numbers

$$\delta = \sum_{i,j=1}^{n} d_{ij} \alpha_i \beta_j,$$

where the d_{ij} range over all rational integers. That this set is an ideal follows immediately, for if ξ is a number of $\mathfrak{S}$,

$$\xi\delta = \sum_{i,j} d_{ij} \xi \alpha_i \beta_j.$$

But

$$\xi\alpha_i = \sum_{h} e_{ih} \alpha_h,$$

where the e_{ih} are rational integers, since $\alpha_1, \cdots, \alpha_n$ form a basis for the ideal $\mathfrak{K}_\alpha$. Thus

$$\xi\delta = \sum_{i,j,h} d_{ij} e_{ih} \alpha_h \beta_j,$$

which is again a member of the set. We shall denote this product ideal by $\mathfrak{K}_\pi$. Note that ideal multiplication is not always commutative.

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* MacDuffee, Transactions of this Society, loc. cit., p. 74.

Du Pasquier* has proved that if M and N are two integral square matrices of order n, not both singular, they possess a greatest common right divisor

$$D = PM + QN,$$

and that every g. c. r. d. of M and N is given by AD where A is an integral matrix of determinant ± 1. Thus the g. c. r. d. has the same latitude of definition as an ideal matrix. The theorem extends readily to n matrices, and du Pasquier shows how D may be readily computed.

3. THEOREM. *If the ideals K_α and K_β are represented by the matrices G_α and G_β respectively, then every matrix representing the product $K_\pi = K_\alpha K_\beta$ is a greatest common right divisor of the n matrices*

$$G_\alpha S(\beta_i), \qquad (i = 1, 2, \cdots, n),$$

and also of the matrices

$$G_\beta \overline{R}(\alpha_i), \qquad (i = 1, 2, \cdots, n),$$

where $\overline{R}$ is the transpose of R.

Suppose that

$$\alpha_i = \sum a_{ij}e_j, \ \beta_i = \sum b_{ij}e_j, \ \pi_i = \sum p_{ij}e_j, \ (i, j = 1, 2, \cdots, n).$$

Since every number of $K_\alpha K_\beta$ is in K_π, there exist in particular rational integers h_{ijk} such that

$$\alpha_i\beta_j = \sum_{k=1}^{n} h_{ijk}\pi_k, \quad (i, j = 1, 2, \cdots, n).$$

That is,

$$\sum_{k,l,m} a_{ik}b_{jl}c_{klm}e_m = \sum_{k,m} h_{ijk}p_{km}e_m.$$

On account of the linear independence of the basal numbers,

$$\sum_{k,l} a_{ik}b_{jl}c_{klm} = \sum_{k} h_{ijk}p_{km}, \quad (i, j, m = 1, 2, \cdots, n),$$

* Vierteljahrsschrift der Naturforschenden Gesellschaft in Zürich, vol. 51 (1906), p. 89.

which can be written in either of the two forms

$$\sum_k a_{rk}\left(\sum_l b_{jl}c_{kls}\right) = \sum_k h_{rjk}p_{ks},$$

$$\sum_l b_{rl}\left(\sum_k a_{ik}c_{kls}\right) = \sum_k h_{irk}p_{ks}.$$

In matric notation the first of these equations is

$$G_\alpha S(\beta_j) = H_{1j}G_\pi, \qquad H_{1j} = (h_{rjs}),$$

and the second is

$$G_\beta \overline{R}(\alpha_i) = H_{2i}G_\pi, \qquad H_{2i} = (h_{irs}).$$

Thus G_π is a common right divisor of the matrices $G_\alpha S(\beta_j)$, and also of the matrices $G_\beta \overline{R}(\alpha_i)$.

Since every number of K_π is in $K_\alpha K_\beta$, there exist rational integers k_{ijh} such that

$$\pi_i = \sum_{j,h} k_{ijh}\alpha_j\beta_h, \qquad (i = 1, 2, \cdots, n).$$

That is,

$$\sum_t p_{it}e_t = \sum_{j,h,l,m,t} k_{ijh}a_{jl}b_{hm}c_{lmt}e_t,$$

whence

$$p_{it} = \sum_{j,h,l,m} k_{ijh}a_{jl}b_{hm}c_{lmt}, \quad (i, t = 1, 2, \cdots, n).$$

This may be written in either of the forms

$$p_{rs} = \sum_{j,h,l} k_{rjh}a_{jl} \sum_m b_{hm}c_{lms},$$

$$p_{rs} = \sum_{j,h,m} k_{rjh}b_{hm} \sum_l a_{jl}c_{lms}.$$

In matric form these equations are, respectively,

$$G_\pi = \sum_h K_{1h}G_\alpha S(\beta_h), \qquad K_{1h} = (k_{rsh}),$$

$$G_\pi = \sum_j K_{2j}G_\beta \overline{R}(\alpha_j), \qquad K_{2j} = (k_{rjs}).$$

Thus G_π is a greatest common right divisor of the matrices $G_\alpha S(\beta_h)$ and also of the matrices $G_\beta \overline{R}(\alpha_j)$.

If the ideals K_α and K_β are non-singular, we may assume that not every α_i and not every β_i is of norm 0. In particular K_α and K_β may have canonical bases, in which case α_1 and β_1 are the smallest positive integers in the respective ideals. Thus not every $S(\beta_i)$ and not every $\overline{R}(\alpha_i)$ is singular. Since G_α and G_β are non-singular matrices, the matrix G_τ is unique up to a left factor A which is an integral matrix of determinant ± 1.

4. *Canonical Form of an Ideal Product.* We shall indicate just one application of the preceding theorem. The problem of obtaining the canonical form of the product ideal from the canonical forms of the factors has been discussed for quadratic fields,* but not for the general case.

By the method of §3 the matrix G_τ is obtained directly by du Pasquier's method when G_α and G_β are non-singular, that is, in the only case considered for algebraic fields. Then by a theorem due to Hermite† we can, by multiplying G_τ on the left by an integral matrix A of determinant ± 1, obtain the canonical matrix

$$\begin{Vmatrix} g_{11} & 0 & 0 & \cdots & 0 \\ g_{12} & g_{22} & 0 & \cdots & 0 \\ g_{13} & g_{23} & g_{33} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ g_{1n} & g_{2n} & g_{3n} & \cdots & g_{nn} \end{Vmatrix}$$

in which every $g_{ij} \geqq 0$, $g_{ii} > 0$, and each element in the ith column below the main diagonal is reduced modulo g_{ii}. Moreover this canonical form is unique.

All the steps in this reduction are mechanical and involve working only with rational integers.

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* W. B. Carver, American Mathematical Monthly, vol. 18 (1911), pp. 81–87.

† Journal für Mathematik, vol. 41 (1851), p. 193.

A CERTAIN MULTIPLE-PARAMETER EXPANSION*

BY H. P. DOOLE

1. *Introduction.* C. C. Camp† has shown the convergence of the expansion of an arbitrary function in terms of the solutions of the systems of equations

$$X_1' + \left(\lambda a_1 - \sum_{j=2}^{n} \mu_j\right) X_1 = 0,$$

$$X_j' + (\lambda a_j + \mu_j) X_j = 0, \quad (j = 2, 3, \cdots, n),$$

where the a_j's are functions of x, with the boundary conditions

$$X_j(-\pi) = X_j(\pi), \qquad (j = 1, 2, \cdots, n).$$

In this paper it is intended to use a differential system in which the n parameters appear in each equation with a function $a_{jk}(x_j)$ multiplying each. The boundary conditions will also be more general.

2. *Expansion and Convergence Proof.* The system of equations

$$X_j' + \left(\sum_{k=1}^{j} \lambda_k a_{jk}(x_j) - \sum_{k=j+1}^{n} \lambda_k a_{jk}(x_j)\right) X_j = 0,$$

(1)

$$X_n' + \sum_{k=1}^{n} \lambda_k a_{nk}(x_n) X_n = 0, \qquad (j = 1, 2, \cdots, n-1),$$

where the $a_{jk}(x_j)$, $(-a \leqq x_j \leqq b)$, are either positive functions or identically zero, with the boundary conditions

$$X_j(-a) = \gamma_j X_j(b), \quad (\gamma_j > 0), \quad (j = 1, 2, \cdots, n), \tag{2}$$

has the solutions

$$X_j = \exp\left\{-\sum_{1}^{j} \lambda_k A_{jk}(x_j) + \sum_{j+1}^{n}{}' \lambda_k A_{jk}(x_j)\right\}, \tag{3}$$

$$(j = 1, 2, \cdots, n),$$

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* Presented to the Society, June 20, 1929.

† American Journal of Mathematics, vol. 50 (1928), p. 259.

where the symbol $\sum_{j+1}^{n}{}'$ is defined to be zero when $j=n$, and $\int a_{jk}(x_j)dx_j = A_{jk}(x_j)$.

From the boundary conditions we obtain the expression

$$\exp\left\{-\sum_{1}^{j}\lambda_k A_{jk}(-a)+\sum_{j+1}^{n}{}'\lambda_k A_{jk}(-a)\right\} = \gamma_j \exp\left\{-\sum_{1}^{j}\lambda_k A_{jk}(b)+\sum_{j+1}^{n}{}'\lambda_k A_{jk}(b)\right\},$$

or

$$\exp\left\{\left(\sum_{1}^{j}\lambda_k A_{jk}-\sum_{j+1}^{n}{}'\lambda_k A_{jk}\right)(a+b)\right\} = \gamma_j \exp\{2\pi m_j i\}, \quad (m_j = 0, \pm 1, \pm 2, \cdots), \tag{4}$$

where

$$\int_{-a}^{b} a_{jk}(x_j)dx_j = (a+b)A_{jk}.$$

Whence, the principal parameter values of

$$\nu_j \equiv \sum_{1}^{j}\lambda_k A_{jk}-\sum_{j+1}^{n}{}'\lambda_k A_{jk} \tag{5}$$

are $\nu_j^* = (\log \gamma_j + 2\pi m_j i)/(a+b)$, the new parameters ν_j being introduced for simplicity. It may be noticed that the ν^*'s lie on lines parallel to the imaginary axis at a distance $\log \gamma_j/(a+b)$.

The system (5) may be solved for λ_k giving

$$\lambda_k = \sum_{j=1}^{n}(-1)^{j+k}\nu_j D(A_{jk})/D(A), \tag{6}$$

where it is assumed that $D(A)\neq 0$, $D(A)$ being the determinant of (5), and $D(A_{jk})$ the minor of A_{jk} in $D(A)$. $D(a)$ and $D(a_{jk})$ are similar determinants, but with each A_{jk} replaced by the corresponding $a_{jk}(x_j)$.

The system adjoint to (1), (2) is

$$-Y_j' + \left(\sum_{k=1}^{j}\lambda_k a_{jk}(x_j)-\sum_{k=j+1}^{n}{}'\lambda_k a_{jk}(x_j)\right)Y_j = 0, \tag{1'}$$

$$\gamma_j Y_j(-a) = Y_j(b), \tag{2'}$$

which has the same characteristic values λ_k^*, ν_j^* as (1), (2).

Let one asterisk designate any one set of parameter values and two asterisks any other set. Consider the two sets of equations, (1) using one asterisk, and (1′) using two. Multiply the first set by Y_j^{**} and the second by $-X_j^*$, add, and integrate from $-a$ to b, obtaining

$$\int_{-a}^{b}(X_j^{*\prime}Y_j^{**} + X_j^*Y_j^{**\prime})dx_j + \sum_{k=1}^{j}(\lambda_k^* - \lambda_k^{**})\int_{-a}^{b} a_{jk}(x_j)X_j^*Y_j^{**}dx_j - \sum_{k=j+1}^{n}{}'(\lambda_k^* - \lambda_k^{**})\int_{-a}^{b} a_{jk}(x_j)X_j^*Y_j^{**}dx_j = 0.$$

Since the first integral vanishes at the limits, and since $\lambda_k^* \neq \lambda_k^{**}$ for at least one value of k, the determinant of the coefficients of the quantities involving λ_k must vanish, giving the conjugacy condition,

$$(7) \qquad \int_{-a}^{b}{}^{(n)} D(a)\prod_{j=1}^{n} X_j^*Y_j^{**}dx_j = 0,$$

where the symbol (n) means an n-fold integral. If the characteristic values in X_j and Y_j all happen to coincide the above integral becomes, since $\prod_{n=1}^{i} X_j^*Y_j^* = 1$,

$$(8) \qquad \int_{-a}^{b}{}^{(n)} D(a)\prod_{j=1}^{n} dx_j = (a+b)^n D(A).$$

Assuming the expansion

$$f(x) = \sum_{m_j=-\infty}^{+\infty} C_{m_j}\prod_{1}^{n} X_j^*, \quad (x \equiv x_1, x_2, \cdots, x_n),$$

where $f(x)$ is made up of a finite number of pieces, each real, continuous, and possessing a continuous partial derivative in each variable, multiply both sides by $D(a)\prod_1^n Y_j^{**}$ and integrate n times from $-a$ to b. On the right, all the terms where the characteristic numbers are different, vanish, and the one where they coincide gives $C_{m_j}(a+b)^n D(A)$. The constant C_{m_j} may thus be determined and the expansion becomes

$$(9) \qquad f(x) = \sum_{m_j=-\infty}^{+\infty}\int_{-a}^{b}{}^{(n)} f(u)D(a)\prod_{1}^{n} X_j^*(x)Y_j^*(u)du_j \Big/ \left[(a+b)^n D(A)\right].$$

Green's function for this system is

$$(10)\quad G \equiv \begin{cases} \prod_1^n [X_j^*(x)Y_j^*(u)/(1-\gamma_j e^{-\nu_j(a+b)})]/D(A), & (u_j < x_j), \\ \prod_1^n [X_j^*(x)Y_j^*(u)/(e^{\nu_j(a+b)}/\gamma_j - 1)]/D(A), & (u_j > x_j), \end{cases}$$

the residue of which with respect to ν_j is

$$\prod_1^n X_j^*(x)\, Y_j^*(u)/[D(A)(a+b)^n].$$

Hence the residue of

$$^{(n)}\int_a^b f(u)D(a)G\prod_1^n du_j$$

is

$$^{(n)}\int_{-a}^{b} f(u)D(a)\prod_1^n X_j^*(x)Y_j^*(u)du_j/[D(A)(a+b)^n]$$

which is one of the terms of the expansion.

To show that the expansion converges, an extension† of Birkhoff's contour integral method will be used. In the evaluation of the contour integrals the following lemma will be useful.

LEMMA.

$$\lim_{m\to\infty} \frac{1}{2\pi i}\int_{\Gamma_m} \frac{|e^{-h(a+b)\nu}|\,d\nu}{(1-\gamma e^{-(a+b)\nu})\nu} = \begin{cases} 0, & (0<h<1), \\ \frac{1}{2}, & (h=0), \\ -\frac{1}{2}, & (h=1), \end{cases}$$

where Γ_m represents any contour which encloses the $(2m+1)$ poles of the integrand, one of which is at the origin with the rest equally spaced on a line parallel to the imaginary axis.

One may use circles with centers at the origin, expanding in such a way that they always pass midway between successive poles, or rectangles of fixed width that include the line of poles and the origin, with ends that recede indefinitely and pass midway between the poles as before.

The contour integral to be evaluated in the convergence proof has 2^n pieces, one of which has the form

† Camp, loc. cit.

$$(11)\qquad I_{-a}^{x} = \lim_{m_j\to\infty} \int_{\Gamma_{m_j}}^{(n)} \int_{-a}^{x}$$

$$\frac{f(u)D(a)\prod_1^n \exp\left\{\int_{x_j}^{u_j}\left[\sum_1^j \lambda_k a_{jk}(x_j) - \sum_{j+1}^{n}{}' \lambda_k a_{jk}(x_j)\right]dx_j\right\} du_j dv_j}{(2\pi i)^n D(A)\prod_1^n (1-\gamma_j e^{-v_j(a+b)})},$$

where the contours Γ_{m_j} are those used in the lemma. The "u" integrals will be evaluated first. $D(a)$ is replaced by the identical expression

$$(1/\lambda_1)\sum_{j=1}^{n}(-1)^{j+1}D(a_{j1})\left[\sum_{k=1}^{j}\lambda_k a_{jk}(x_j) - \sum_{k=j+1}^{n}{}'\lambda_k a_{jk}(x_j)\right]$$

and each term of the summation with the exponential factor containing the corresponding variable is integrated by parts giving

$$(12)\quad \left\{\sum_{j=1}^{n}(-1)^{j+1}\int_{-a}^{(n-1)}{}^{x^j}\left[f(x_j,u^j) - f(-a,u^j)\exp\left\{\int_{x_j}^{-a}\left[\sum_1^j \lambda_k a_{jk}(x_j) - \sum_{j+1}^{n}{}'\lambda_k a_{jk}(x_j)\right]dx_j\right\}\right]\right.$$

$$\cdot D(a_{j1})\prod_{l=1}^{n}\exp\left\{\int_{x_l}^{u_l}\left[\sum_{k=1}^{l}\lambda_k a_{lk}(x_l) - \sum_{k=l+1}^{n}{}'\lambda_k a_{lk}(x_l)\right]dx_l\right\}du_l$$

$$-\int_{-a}^{(n)}{}^{x}\left[\sum_{j=1}^{n}(-1)^{j+1}\frac{\partial f(u)}{\partial u_j}D(a_{j1})\right]\prod_1^n \exp\left\{\int_{x_j}^{u_j}\left[\sum_1^j \lambda_k a_{jk}(x_j)\right.\right.$$

$$\left.\left.\left. - \sum_{j+1}^{n}{}'\lambda_k a_{jk}(x_j)\right]dx_j\right\}du_j\right\}/[D(A)\lambda_1]$$

where the superscript j means that the variable with the corresponding subscripts is missing, that is, $x^j \equiv x_1, x_2, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n$. Thus $f(x_2, u^2) \equiv f(u_1, x_2, u_3, \cdots, u_n)$. A superscript with a $\sum$ or $\prod$ sign has a similar meaning.

To continue the integration of the terms involving $f(x_j, u^j)$, the first minors, $D(a_{j1})$, are developed according to rows in terms of second minors, thus making $n-1$ terms in place of each. As each row in $D(a)$ contains only the variables x_j, each of the

$n-1$ terms obtained from $D(a_{j1})$ consists of $a_{j'k}(x_{j'})$ and a second minor independent of $x_{j'}$. In this integration j' is different from the j in the original integration. In the exponential factor of $a_{j'k}(x_{j'})$ that involves $x_{j'}$, replace λ_k by its value obtained by solving the jth equation of (5). The exponent will be

$$\int_{x_{j'}}^{u_{j'}} \left[\nu_{j'} a_{j'k}(x_{j'})dx_{j'} - \left(\sum_{q=1}^{j'} - \sum_{q=j'+1}^{n}{}' \right)^{k} \lambda_q \begin{vmatrix} a_{j'k}(x_{j'}) & a_{j'q}(x_{j'}) \\ A_{j'k} & A_{j'q} \end{vmatrix} dx_{j'} \right] \Big/ A_{j'k} .$$

If in an integration by parts,

$$a_{j'k}(x_{j'}) \exp \left\{ \int_{x_{j'}}^{u_{j'}} \nu_{j'} a_{j'k}(x_{j'})dx_{j'}/A_{j'k} \right\} du_{j'}$$

is taken as the "dv" factor, the upper limit term of the integrated part will be $f(x_j, x_{j'}, u^{j,j'})A_{j'k}/\nu_{j'}$, j' being any j different from the former j, with a similar expression for each of the $n-1$ terms. Continuing this process by developing the second minors by rows and third minors, integrating by parts again with respect to a different x_j, and repeating until all n integrations have been performed, we find that the expression $f(x)D(A_{j1})/\prod_{l=1}^{n}\nu_l$ results from each of the terms involving $f(x_j, u^j)D(a_{j1})$. Then by adding the n similar terms obtained from the whole summation, and dividing by the factor $\lambda_1 D(A)$, we obtain the expression $f(x)/\prod_1^n \nu_j$.

The n-fold contour integration of this expression by means of the lemma gives the result $f(x)/2^n$. The other 2^n-1 "I" integrals give similar results, the combination of which form the so-called mean value of $f(x)$. If the lower limit terms and other integrals that arise in the various integrations by parts can be shown to contribute zero in the limit, the convergence of the expansion may be considered proved.

The integrals of the lower limit terms of (12) may be evaluated by comparing them with a simpler series obtained by the following method. First, replace each $f(-a, u^j)$ by a common upper bound, and expand each $D(a_{j1})$ by any row and second minors. Next replace each λ_k in the exponential factor of $a_{j'k}(x_{j'})$ that involves $x_{j'}$ by its value obtained from the jth equation of (5) as in the preceding integration. Then take absolute values and remove the exponential factors that still

contain λ's from the integrals by the first mean value theorem. The integration with respect to $x_{j'}$ may now be performed. Repeat the process by expanding the second minors by rows into third minors, and so on until $n-1$ integrations have been performed. To carry out the contour integration put all resulting terms over the same denominator $\prod_1^n \nu_j$, and remove from each term the exponential factors involving λ's and the factors $\nu_j/[D(A)\lambda_1]$ all of which are bounded on the rectangular contours mentioned in the lemma. The remaining exponentials are of the form

$$\exp\left\{-\nu_j \int_{-a}^{x_j} a_{jk}(x_j)dx_j/A_{jk}\right\} = \exp\{-\nu_j h(a+b)\}, \qquad (0 < h < 1),$$

and the contour integrals accordingly contribute zero in the limit. All the lower limit terms that arise in the other integrations by parts may be treated similarly.

If $f(x)$ is only piecewise (stückweise) continuous, at each point of discontinuity certain terms will occur in the integration by parts which contribute zero to the final result after a contour integration similar to that used for the lower limit terms.

In the various integrations by parts, the unintegrated terms will have the form

$$-\int_{-a}^{x_{j'}}\left[\frac{\partial f(x_j, u^j)}{\partial u_j}\right.$$

$$\left.-\left(\sum_{q=1}^{j'} - \sum_{q=j'+1}^{n}{}'\right)^k \lambda_q \begin{vmatrix} a_{j'k}(x_{j'}) & a_{j'q}(x_{j'}) \\ A_{j'k} & A_{j'q} \end{vmatrix} \Big/ A_{j'k}\right]\frac{A_{j'k}}{\nu_{j'}}$$

$$\cdot\exp\left\{\int_{z_{j'}}^{u_{j'}}\left[\nu_{j'}a_{j'k}(x_{j'})\right.\right.$$

$$\left.\left.-\left(\sum_{q=1}^{j'} - \sum_{q=j'+1}^{n}{}'\right)^k \lambda_q \begin{vmatrix} a_{j'k}(x_{j'}) & a_{j'q}(x_{j'}) \\ A_{j'k} & A_{j'q} \end{vmatrix}\right]dx_{j'}/A_{j'k}\right\}du_{j'}$$

with the remaining exponentials and a second minor as factors in the $(n-1)$-fold integral. To integrate, replace the factor in the bracket by an upper bound, take absolute values and remove the exponential by the first mean value theorem, obtaining

$$-A_{j'k}\exp\left\{\int_{x_{j'}}^{\eta}(\cdots)dx_{j'}\right\}(a+x_{j'})/\nu_{j'},\qquad (u_{j'}<\eta<x_{j'}),$$

as a result. For the other $n-2$ integrations expand the second minors by rows and third minors, remove exponential factors containing λ's by the first mean value theorem, and proceed as before. The contour integrals will then approach zero in the limit.

To determine the contribution of the last integral in (12), replace each $\partial f(u)/\partial u_j$ by a common upper bound, expand each $D(a_{j1})$ by rows and second minors, take absolute values and proceed as for the lower limit terms for $n-1$ integrations. To make the nth integration, remove the exponential by the first mean value theorem and integrate directly Then, if we remove bounded factors, the contour integration follows as before and contributes zero in the limit. We have thus proved the following theorem.

THEOREM. *Let $f(x)$ be made up of a finite number of pieces, each real, continuous, and possessing a continuous partial derivative in each argument in the region $-a\leqq x_j\leqq b$, while each $a_{jk}(x_j)$ is integrable and either positive or identically zero over the region. Then the expansion $\sum_{m_j}C_{m_j}\prod_1^n X_j^*$, $(m_j=0,\ \pm1,\ \pm2,\cdots)$, where X_j^* is a characteristic solution of the jth equation of* (1), (2), *and where*

$$C_{m_j}=\int_{-a}^{(n)}{}^{b}f(u)D(a)\prod_1^n Y_j^*(u)du_j/\left[(a+b)^n D(A)\right],$$

converges to the so-called mean value of $f(x)$ at any interior point of the region.

THE UNIVERSITY OF NEBRASKA

PROJECTIVE INTEGRAL INVARIANTS ATTACHED TO THE TRAJECTORIES OF DIFFERENTIAL SYSTEMS*

BY A. D. MICHAL

1. *Introduction.* In his book on integral invariants, E. Cartan† showed how to every integral invariant in the sense of Poincaré there corresponds a more general integral invariant (called *complete* by Cartan) in an associated manifold of one more dimension. It turns out, as Goursat‡ has more recently shown, that a complete integral invariant is a Poincaré integral invariant attached to the trajectories of an associated differential system.

Cartan, however, did not give necessary and sufficient conditions; neither did he go into the question of the existence of other integral invariants in the same associated manifold. The present paper is concerned with such problems. As in my previous contributions§ to the subject of integral invariants, the tensor methods and the language of group theory will be employed throughout the paper.

2. *Projective Transformations of Coordinates and Projective Tensors.* In this paragraph and throughout our whole paper we shall understand that *a Greek index can take on any of the values* $0, 1, 2, \cdots, n$, *while a Latin index can take on any of the values* $1, 2, \cdots, n$. The repetition of an index in a term will be used to denote summation with respect to that index over all its admissible values.

Let $\xi^i(x^1, \cdots, x^n)$ be a contravariant vector in an n-dimensional manifold $(x^1, x^2, \cdots, x^n)$. Consider an associated $(n+1)$-dimensional manifold with points having coordinates $(x^0, x^1, \cdots, x^n)$ and subject to the analytic transformations

$$(1) \qquad \mathfrak{G}: \begin{aligned} \bar{x}^i &= f^i(x^1, x^2, \cdots, x^n), \\ \bar{x}^0 &= x^0 + f^0(x^1, x^2, \cdots, x^n), \end{aligned} \qquad \left|\frac{\partial f^i}{\partial x^j}\right| \neq 0,$$

* Presented to the Society, November 29, 1930.

† E. Cartan, *Leçons sur les Invariants Intégraux*, 1922.

‡ E. Goursat, Comptes Rendus, vol. 174 (1922), pp. 1089–1091.

§ A. D. Michal, Transactions of this Society, vol. 29 (1927), pp. 612–646.

where $f^0(x^1, \cdots, x^n)$ satisfies the differential equation

$$\xi^i \frac{\partial f}{\partial x^i} = 0. \tag{2}$$

THEOREM 1. *The differential system*

$$\frac{dx^\alpha}{d\tau} = \xi^\alpha(x^1, \cdots, x^n), \qquad (\xi^0 \equiv 1), \tag{3}$$

remains invariant in form under the coordinate transformations $\mathfrak{G}$ *in* $x^0, x^1, \cdots, x^n$.

In fact the functions $\bar{\xi}^\alpha$ in the transformed equations

$$\frac{d\bar{x}^\alpha}{d\tau} = \bar{\xi}^\alpha$$

are given by

$$\begin{cases} \bar{\xi}^i = \xi^j \dfrac{\partial \bar{x}^i}{\partial x^j} + \dfrac{\partial \bar{x}^i}{\partial x^0}, \\ \bar{\xi}^0 = \xi^j \dfrac{\partial \bar{x}^0}{\partial x^j} + \xi^0 \dfrac{\partial \bar{x}^0}{\partial x^0}. \end{cases} \tag{4}$$

But

$$\frac{\partial \bar{x}^i}{\partial x^0} = 0, \quad \frac{\partial \bar{x}^0}{\partial x^0} = 1 \text{ and } \xi^j \frac{\partial \bar{x}^0}{\partial x^j} = 0.$$

The truth of the theorem is now immediate.

One can also verify that the transformations $\mathfrak{G}$ are the most general coordinate transformations in $x^0, x^1, \cdots, x^n$ that leave the differential equations (3) invariant in form.

We shall use the term *projective tensor** to denote a tensor whose components depend on $x^1, x^2, \cdots, x^n$ but not on x^0 and which is subject to the coordinate transformations $\mathfrak{G}$. It is obvious from the definition that the transformed components of a projective tensor depend only on $\bar{x}^1, \bar{x}^2, \cdots, \bar{x}^n$ and not on $\bar{x}^0$. As examples we may take the projective contravariant and covariant vectors

* Compare with T. Y. Thomas' use of the same terminology in another connection. See his paper in Mathematische Zeitschrift, vol. 25 (1926), pp. 723–733.

$$\overline{A}^{\alpha}(\bar{x}^1, \cdots, \bar{x}^n) = A^{\sigma}(x^1, \cdots, x^n)\frac{\partial \bar{x}^{\alpha}}{\partial x^{\sigma}} \rightarrow \begin{cases} \overline{A}^{i} = A^{j}\dfrac{\partial \bar{x}^{i}}{\partial x^{j}}, \\ \overline{A}^{0} = A^{j}\dfrac{\partial \bar{x}^{0}}{\partial x^{j}} + A^{0}; \end{cases}$$

$$\overline{A}_{\alpha}(\bar{x}^1, \cdots, \bar{x}^n) = A_{\sigma}(x^1, \cdots, x^n)\frac{\partial x^{\sigma}}{\partial \bar{x}^{\alpha}} \rightarrow \begin{cases} \overline{A}_{i} = A_{j}\dfrac{\partial x^{j}}{\partial \bar{x}^{i}} + A_{0}\dfrac{\partial x^{0}}{\partial \bar{x}^{i}}, \\ \overline{A}_{0} = A_{0}. \end{cases}$$

3. *Some Preliminary Theorems on Integral Invariants.* I shall summarize in this paragraph a few theorems that will be found useful in the succeeding paragraphs. These theorems are special cases of some more general theorems proved by me in another paper.*

THEOREM 2. *A necessary and sufficient condition that*

$$A_i \delta x^i \tag{5}$$

be an integral invariant of the one-parameter group

$$\frac{dx^i}{d\tau} = \xi^i(x^1, \cdots, x^n) \tag{6}$$

is that the functions $A_i(x^1, \cdots, x^n)$ satisfy the system of tensor differential equations

$$\left(\frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i}\right)\xi^j + \frac{\partial (A_j \xi^j)}{\partial x^i} = 0. \tag{7}$$

The corresponding tensor differential equations for the invariance of the alternating form

$$A_{ij}(x^1, \cdots, x^n)\delta_1 x^i \delta_2 x^j \quad (A_{ij} = -A_{ji}) \tag{8}$$

are given by

$$\left(\frac{\partial A_{ij}}{\partial x^k} - \frac{\partial A_{kj}}{\partial x^i} - \frac{\partial A_{ik}}{\partial x^j}\right)\xi^k + \frac{\partial (A_{il}\xi^l)}{\partial x^j} - \frac{\partial (A_{jl}\xi^l)}{\partial x^i} = 0. \tag{9}$$

THEOREM 3. *A necessary and sufficient condition that* (5) *be an integral invariant that is attached*† *to the path curves of the group* (6) *is that the functions A_i satisfy the tensor equations*

* A. D. Michal, loc. cit.

† E. Goursat, *Leçons sur le Probleme de Pfaff*, 1922, p. 236.

$$\left(\frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i}\right)\xi^j = 0, \quad A_i\xi^i = 0. \tag{10}$$

The corresponding tensor equations for the form (8) *are*

$$\left(\frac{\partial A_{ij}}{\partial x^k} - \frac{\partial A_{kj}}{\partial x^i} - \frac{\partial A_{ik}}{\partial x^j}\right)\xi^k = 0, \quad A_{ij}\xi^j = 0. \tag{11}$$

4. *Linear Integral Invariants.* One can easily show that a necessary and sufficient condition that a differential form

$$A_\alpha(x^1, \cdots, x^n)\delta x^\alpha \tag{12}$$

be an absolute form under the coordinate transformations $\mathfrak{G}$ is that $A_0, A_1, \cdots, A_n$ be the components of a projective covariant vector. This is the reason why we shall call the differential form (12) *a projective differential form.* Similar remarks hold good for the projective differential form

$$A_{\alpha\beta}\delta_1 x^\alpha \delta_2 x^\beta \qquad (A_{\alpha\beta} = -A_{\beta\alpha}), \tag{13}$$

THEOREM 4. *A necessary and sufficient condition that the projective differential form*

$$A_\alpha(x^1, \cdots, x^n)\delta x^\alpha \tag{14}$$

be an invariant of the group

$$\frac{dx^\alpha}{d\tau} = \xi^\alpha(x^1, \cdots, x^n) \qquad (\xi^0 \equiv 1), \tag{15}$$

is that the form $A^i(x^1, \cdots, x^n)\delta x^i$ *be an invariant of* (6) *and that* $A_0(x^1, \cdots, x^n)$ *be a scalar invariant of* (6).

An application of Theorem 2 to the case in hand yields the conditions

$$\left(\frac{\partial A_\alpha}{\partial x^\beta} - \frac{\partial A_\beta}{\partial x^\alpha}\right)\xi^\beta + \frac{\partial(A_\sigma\xi^\sigma)}{\partial x^\alpha} = 0. \tag{16}$$

Since A_α and ξ^α are independent of x^0 and $\xi^0 \equiv 1$, it follows by a simple calculation that the conditions (16) split up into the conditions (7) and

$$\frac{\partial A_0}{\partial x^i}\xi^i = 0. \tag{17}$$

But (7) are the conditions for the invariance of $A_i\delta x^i$ under the group (6) while (17) is, by a well known theorem of Lie, a necessary and sufficient condition that $A_0(x^1, \cdots, x^n)$ be a scalar invariant of the group (6).

THEOREM 5. *A necessary and sufficient condition that the projective differential form* (14) *be an invariant attached to the path curves of the group* (15) *is that $A_i\delta x^i$ be an invariant of* (6) *and that*

$$A_0(x^1, \cdots, x^n) = -A_k\xi^k. \tag{18}$$

With the aid of Theorem 3 we obtain the necessary and sufficient conditions

$$\left(\frac{\partial A_\alpha}{\partial x^\beta} - \frac{\partial A_\beta}{\partial x^\alpha}\right)\xi^\beta = 0, \quad A_\sigma\xi^\sigma = 0. \tag{19}$$

The first set of equations in (19) yields the conditions

$$\left(\frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i}\right)\xi^j = \frac{\partial A_0}{\partial x^i}, \quad \frac{\partial A_0}{\partial x^j}\xi^j = 0 \tag{20}$$

while the last equation obviously is equivalent to

$$A_0 = -A_k\xi^k. \tag{21}$$

But the equations (20) and (21) are completely equivalent to the conditions (7) and (21). This completes the proof of our theorem.

It is interesting to observe that the function A_0, as given by (18), is ipso facto a scalar invariant* of the group (6). Hence the attached invariant is distinguished from the unattached invariant (considered in Theorem 4) by the fact that not only must A_0 be a scalar invariant of the group (6) but it must have the particular form (18).

THEOREM 6. *There always exist projective invariants of the types dealt with in Theorems*† 4 *and* 5 *for any chosen one-parameter group appropriate to the case.*

Cartan‡ has given a number of interesting examples of projective invariants of the type considered in Theorem 5. He did

* E. Goursat, loc. cit., p. 255.

† The same statement holds good for the invariants of Theorems 7 to 11.

‡ E. Cartan, loc. cit.

not, however, give necessary and sufficient conditions nor did he write the conditions in an arbitrary coordinate system. Cartan referred to such invariants as complete integral invariants. Similar remarks hold good for the corresponding projective bilinear forms that are treated in the next paragraph.

5. *Alternating Integral Invariants.*

THEOREM 7. *A necessary and sufficient condition that the projective differential form*

$$A_{\alpha\beta}(x^1, \cdots, x^n)\delta_1 x^\alpha \delta_2 x^\beta, \qquad (A_{\alpha\beta} = -A_{\beta\alpha}), \tag{22}$$

be an invariant of the group (15) *is that both* $A_{ij}\delta_1 x^i \delta_2 x^j$ *and* $A_{0j}\delta x^j$ *be invariants of the group* (6).

With the aid of the second part of Theorem 2 we see that necessary and sufficient conditions for the invariance of (22) are given by the system of differential equations

$$\left(\frac{\partial A_{\alpha\beta}}{\partial x^\gamma} - \frac{\partial A_{\gamma\beta}}{\partial x^\alpha} - \frac{\partial A_{\alpha\gamma}}{\partial x^\beta}\right)\xi^\gamma + \frac{\partial(A_{\alpha\lambda}\xi^\lambda)}{\partial x^\beta} - \frac{\partial(A_{\beta\lambda}\xi^\lambda)}{\partial x^\alpha} = 0. \tag{23}$$

For $\alpha = i$, $\beta = j$, these equations reduce to

$$\left(\frac{\partial A_{ij}}{\partial x^k} - \frac{\partial A_{kj}}{\partial x^i} - \frac{\partial A_{ik}}{\partial x^j}\right)\xi^k + \frac{\partial(A_{il}\xi^l)}{\partial x^j} - \frac{\partial(A_{jl}\xi^l)}{\partial x^i} = 0, \tag{24}$$

while for $\alpha = 0$, $\beta = j$, they reduce to

$$\left(\frac{\partial A_{0j}}{\partial x^k} - \frac{\partial A_{0k}}{\partial x^j}\right)\xi^k + \frac{\partial(A_{0k}\xi^k)}{\partial x^j} = 0. \tag{25}$$

The remaining equations in (23) are automatically satisfied on the basis of (24) and (25). Hence our theorem follows by an application of Theorem 2.

THEOREM 8. *A necessary and sufficient condition that the projective form* (22) *be an invariant attached to the path curves of the one-parameter group* (15) *is that* $A_{ij}\delta_1 x^i \delta_2 x^j$ *be an invariant of the group* (6) *and that* $A_{i0}(x^1, \cdots, x^n)$ *be given by*

$$A_{i0} = -A_{ik}\xi^k. \tag{26}$$

With the aid of the second part of Theorem 3, we obtain the necessary and sufficient conditions

$$\left(\frac{\partial A_{\alpha\beta}}{\partial x^\gamma} - \frac{\partial A_{\gamma\beta}}{\partial x^\alpha} - \frac{\partial A_{\alpha\gamma}}{\partial x^\beta}\right)\xi^\gamma = 0, \quad A_{\alpha\lambda}\xi^\lambda = 0. \tag{27}$$

As can be readily verified, these equations split up into the following set of equations

$$\begin{cases} \left(\dfrac{\partial A_{ij}}{\partial x^k} - \dfrac{\partial A_{kj}}{\partial x^i} - \dfrac{\partial A_{ik}}{\partial x^j}\right)\xi^k = \dfrac{\partial A_{i0}}{\partial x^j} - \dfrac{\partial A_{j0}}{\partial x^i}, \\ \left(\dfrac{\partial A_{0j}}{\partial x^k} - \dfrac{\partial A_{0k}}{\partial x^j}\right)\xi^k = 0, \\ A_{i0} = -A_{ik}\xi^k, \quad A_{0k}\xi^k = 0. \end{cases} \tag{28}$$

The second set of equations is automatically satisfied on the basis of the first set and the last equation on the basis of the third set. Hence (28) is completely equivalent to the system of equations (9) and (26). This proves our theorem.

Definition. A linear differential form ω_δ will be said to be a *relative integral invariant* of a one-parameter group appropriate to the case whenever the bilinear covariant $\delta_2\omega_{\delta_1} - \delta_1\omega_{\delta_2}$ is an invariant of the same one-parameter group.

THEOREM 9. *A necessary and sufficient condition that $A_\alpha\delta x^\alpha$ be a relative projective integral invariant of the group* (15) *is that $A_i\delta x^i$ be a relative integral invariant of* (6) *and that A_0 satisfy the differential equation*

$$\frac{\partial A_0}{\partial x^k}\xi^k = K, \tag{29}$$

where K is any chosen constant.

In fact, with the aid of Theorem 7 one can deduce without much difficulty the necessary and sufficient conditions

$$\begin{cases} \dfrac{\partial(A_{ik}\xi^k)}{\partial x^j} - \dfrac{\partial(A_{jk}\xi^k)}{\partial x^i} = 0, \\ \left(\dfrac{\partial A_{0j}}{\partial x^k} - \dfrac{\partial A_{0k}}{\partial x^j}\right)\xi^k + \dfrac{\partial(A_{0k}\xi^k)}{\partial x^j} = 0, \end{cases} \tag{30}$$

where $A_{\alpha\beta} = \partial A_\alpha/\partial x^\beta - \partial A_\beta/\partial x^\alpha$. Now the second set of equations (30) reduces to

$$\frac{\partial}{\partial x^i}\left(\frac{\partial A_0}{\partial x^k}\xi^k\right) = 0,$$

which proves our theorem.

THEOREM 10. *A necessary and sufficient condition that* $A_\alpha \delta x^\alpha$ *be a relative projective invariant attached to the path curves of the group* (15) *is that**

(a) *the Pfaffian equation*

$$dA_0 = \left(\frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i}\right)\xi^j dx^i \tag{31}$$

be completely integrable;

(b) A_0 *satisfy the equation* (31).

We leave the details of proof to the reader for lack of space.

6. *Concluding Remarks.* The preceding paragraph was concerned with alternating projective differential forms and the reader may now wonder whether there are corresponding theorems for symmetric projective forms

$$g_{\alpha\beta}(x^1, \cdots, x^n)\delta x^\alpha \delta x^\beta, \qquad (g_{\alpha\beta} = g_{\beta\alpha}). \tag{32}$$

There is a partial negative answer to this query, for there does not exist a non-degenerate projective form (32) (that is, a form with a non-vanishing discriminant) attached to the path curves of the group (15). However, the following theorem holds good.

THEOREM 11. *A necessary and sufficient condition that* (32) *be a projective invariant of the group* (15) *is that* $g_{ij}\delta x^i \delta x^j$ *and* $g_{i0}\delta x^i$ *be invariants of the group* (6) *and that* g_{00} *be a scalar invariant of* (6).

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* As an interesting application we observe that for the case of the Hamilton canonical equations, the Hamiltonian $H(q, p)$ is the unique function, up to an additive constant, in the relative invariant $p_i\delta q + A_0\delta t$.

A NOTE ON GEOMETRICAL FACTORIAL SERIES

BY G. W. STARCHER

1. *Introduction.* This note is concerned with series of the form

$$c_0 + \sum_{\nu=1}^{\infty} \frac{c_\nu}{(x-q)(x-q^2)\cdots(x-q^\nu)}, \tag{1}$$

where q is such that $|q|<1$. Series of this form are called geometrical factorial series. By comparing (1) with the descending power series

$$c_0 + \sum_{\nu=1}^{\infty} c_\nu x^{-\nu} \tag{2}$$

it is shown that they have precisely the same points of convergence, and hence a number of fundamental properties of (1) follow from the known properties of the descending power series. In §3 a fundamental theorem concerning the representation of analytic functions by means of series of the form (1) is proved. In §4 precise formulas for the multiplication of two such series are obtained. The points $x=q^i$, $(i=1, 2, 3, \cdots)$, will be called exceptional points for the series (1), and we agree to exclude such points from consideration. If we replace x by x^{-1}, the series (1) becomes

$$c_0 + \sum_{\nu=1}^{\infty} \frac{c_\nu x^\nu}{(1-qx)(1-q^2x)\cdots(1-q^\nu x)},$$

which, if it converges at all, converges for values of x in the neighborhood of the origin and defines an analytic function in such a region. Such series might be compared to the ascending power series. This discussion is presented with reference to the series (1) only because of convenience and simplicity of formulas.

2. *Comparison of* (1) *and* (2). Suppose x to be a point of convergence of (2). Let us write

$$u_\nu = c_\nu x^{-\nu}, \quad v_\nu = \frac{x^\nu}{(x-q)(x-q^2)\cdots(x-q^\nu)}.$$

Then

$$|v_{\nu+1} - v_\nu| = \left|\left\{\frac{x^\nu}{(x-q)(x-q^2)\cdots(x-q^\nu)}\right\}\left\{\frac{x}{x-q^{\nu+1}} - 1\right\}\right|$$

$$= \left|\frac{q^{\nu+1}x^\nu}{(x-q)(x-q^2)\cdots(x-q^{\nu+1})}\right|.$$

By the ratio test for convergence we can show that the series $\sum_{\nu=1}^{\infty}|v_{\nu+1}-v_\nu|$ converges for every x not already excluded. By a known theorem* we have the result that

$$\sum_{\nu=0}^{\infty} u_\nu v_\nu = c_0 + \sum_{\nu=1}^{\infty} \frac{c_\nu}{(x-q)(x-q^2)\cdots(x-q^\nu)}$$

converges for all values of x for which (2) converges.

Conversely, let x be a point of convergence of the series (1). Let

$$u_\nu = \frac{c_\nu}{(x-q)(x-q^2)\cdots(x-q^\nu)},$$

$$v_\nu = \frac{(x-q)(x-q^2)\cdots(x-q^\nu)}{x^\nu}.$$

It is easy to show by the ratio test that the series $\sum_{\nu=1}^{\infty}|v_{\nu+1}-v_\nu|$ converges for every x, and hence the series

$$\sum_{\nu=1}^{\infty} u_\nu v_\nu = c_0 + \sum_{\nu=1}^{\infty} c_\nu x^{-\nu}$$

converges at the points of convergence of (1). It follows that the theorems concerning the region of convergence and expressions for the radius of convergence known for the power series (2) also apply to the series (1). By means of the second lemma in the paper already cited we can establish the corresponding theorems for uniform convergence when we exclude the exceptional points by enclosing them in arbitrarily small circles.

3. *Representation of Functions by Series of the Form* (1). We shall establish the following theorem.

THEOREM. *Let $f(x)$ be analytic outside the circle S of radius R about the origin. Then $f(x)$ can be represented uniquely by a series of the form*

* See, for example, Carmichael, Transactions of this Society, vol. 17 (1916), p. 211.

$$c_0 + \sum_{\nu=1}^{\infty} \frac{c_\nu}{(x-q)(x-q^2)\cdots(x-q^\nu)}, \qquad |q| < 1,$$

where

$$c_k = \frac{1}{2\pi i}\int_C (z-q)(z-q^2)\cdots(z-q^{k-1})f_0(z)dz, \qquad k > 0,$$

$f_0(z)$ being written for $f(z)-c_0$. The contour C is understood to be any contour about the origin lying entirely outside S, such that the point x lies outside C, and C does not pass through any exceptional points.

Let $f(\infty)=c_0$. Consider the function $f_0(x)=f(x)-c_0$. By the Cauchy integral theorem we have*

$$f_0(x) = -\frac{1}{2\pi i}\int_C \frac{f_0(z)dz}{z-x} + \frac{1}{2\pi i}\int_{C'} \frac{f_0(z)dz}{z-x},$$

where C and C' are circles lying outside S and are such that the point x lies inside C' but is outside C. By letting the radius of C' increase indefinitely we can show that the second integral contributes zero, since $\lim_{z\to\infty} f_0(z)=0$, and we have finally

$$f_0(x) = \frac{1}{2\pi i}\int_C \frac{f_0(z)dz}{x-z}. \tag{3}$$

We can easily verify the following algebraic identity:

$$\frac{1}{x-z} = \frac{1}{x-q^s} + \frac{z-q^s}{x-q^s}\cdot\frac{1}{x-z}. \tag{4}$$

If in this expression we first take $s=1$, then replace $1/(x-z)$ in the second member by its value above when $s=2$, again when $s=3$, and continue k times, we have the algebraic identity

$$\begin{aligned}\frac{1}{x-z} &= \frac{1}{x-q} + \frac{z-q}{x-q}\cdot\frac{1}{x-z} \\ &= \frac{1}{x-q} + \frac{z-q}{(x-q)(x-q^2)} + \frac{(z-q)(z-q^2)}{(x-q)(x-q^2)}\cdot\frac{1}{x-z} \\ &= \frac{1}{x-q} + \sum_{j=1}^{k-1}\frac{(z-q)(z-q^2)\cdots(z-q^j)}{(x-q)(x-q^2)\cdots(x-q^{j+1})} \\ &\qquad + \frac{(z-q)\cdots(z-q^k)}{(x-q)\cdots(x-q^k)}\cdot\frac{1}{x-z}.\end{aligned} \tag{5}$$

* See Whittaker and Watson, *Modern Analysis*, 4th ed., §5.21.

Then by (3) and (5) we have

$$
\begin{aligned}
f_0(x) &= \frac{1}{2\pi i}\int_C \frac{f_0(z)dz}{x-z} \\
&= \sum_{i=0}^{k-1} \frac{1}{2\pi i}\int_C \frac{(z-q)(z-q^2)\cdots(z-q^i)f_0(z)}{(x-q)(x-q^2)\cdots(x-q^{i+1})}\,dz \\
&\quad + \frac{1}{2\pi i}\int_C \frac{(z-q)(z-q^2)\cdots(z-q^k)f_0(z)}{(x-q)(x-q^2)\cdots(x-q^k)(x-z)}\,dz.
\end{aligned} \tag{6}
$$

Since z lies on C we have the following inequality, $|z|/|x| \leqq \rho < 1$. We denote by R_k the last term in (6). For every positive number ϵ there exists an integer k_0 such that for $k > k_0$ we can write

$$\left|\frac{z-q^k}{x-q^k}\right| < \rho + \epsilon.$$

Then we have

$$
\begin{aligned}
&\left|\frac{1}{2\pi i}\int_C \frac{(z-q)(z-q^2)\cdots(z-q^{k_0})(z-q^{k_0+1})\cdots(z-q^k)f_0(z)}{(x-q)(x-q^2)\cdots(x-q^{k_0})(x-q^{k_0+1})\cdots(x-q^k)(x-z)}dz\right| \\
&\qquad \leqq \frac{M}{2\pi}\left|\int_C (\rho+\epsilon)^{k-k_0}dz\right| = M(\rho+\epsilon)^{k-k_0}, \quad k > k_0.
\end{aligned}
$$

When ϵ is sufficiently small, $\rho+\epsilon<1$ and $\lim_{k\to\infty} M(\rho+\epsilon)^{k-k_0}=0$. That is, $\lim_{k\to\infty} R_k=0$. From (6) we have

$$
\begin{aligned}
f_0(x) &= \frac{1}{2\pi i}\int_C \frac{f_0(z)dz}{x-q} \\
&\quad + \sum_{r=2}^{\infty} \frac{1}{2\pi i}\int_C \frac{(z-q)(z-q^2)\cdots(z-q^{r-1})f_0(z)}{(x-q)(x-q^2)\cdots(x-q^r)}dz.
\end{aligned} \tag{7}
$$

This series converges and represents the analytic function $f_0(x)$ outside the circle S. Then we have

$$f(x) = c_0 + f_0(x) = c_0 + \sum_{r=1}^{\infty} \frac{c_r}{(x-q)(x-q^2)\cdots(x-q^r)}, \tag{8}$$

where

$$c_k = \frac{1}{2\pi i}\int_C (z-q)(z-q^2)\cdots(z-q^{k-1})f_0(z)dz, \quad k > 0,$$

as the theorem states.

Finally, there can be but one such expansion. For suppose we also had

$$(9) \qquad f(x) = d_0 + \sum_{r=1}^{\infty} \frac{d_r}{(x-q)(x-q^2)\cdots(x-q^r)}.$$

Subtracting (9) from (8) member by member gives

$$0 = c_0 - d_0 + \sum_{r=1}^{\infty} \frac{c_r - d_r}{(x-q)(x-q^2)\cdots(x-q^r)}.$$

Considering the limit approached by this expression as x becomes infinite we have $c_0 - d_0 = 0$, or $c_0 = d_0$. Then

$$0 = \sum_{r=1}^{\infty} \frac{c_r - d_r}{(x-q)(x-q^2)\cdots(x-q^r)}.$$

Multiplying through by $x-q$ gives

$$0 = c_1 - d_1 + \sum_{r=2}^{\infty} \frac{c_r - d_r}{(x-q^2)(x-q^3)\cdots(x-q^r)}.$$

Again letting x become infinite we have $c_1 - d_1 = 0$, or $c_1 = d_1$. Continuing this process we have $c_i = d_i$, $i = 1, 2, 3, \cdots$, and the expansion (8) is unique.

4. *Operations with Series of the Form* (1). Clearly the addition of two series of the form (1) presents no difficulty. That the function represented by the product of two series of the form (1) can be expressed in the same form follows from the properties of the series given in the preceding sections of this paper. We are here concerned with obtaining the precise coefficients in such a product.

In (5) let us write q for z, q^{r+1} for q, then again apply (5) to the second member of the identity after writing q^{r+2} for q, and continue k times. If we denote the last term in the second member by L_k, then $\lim_{k\to\infty} L_k = 0$ and we can write

$$\frac{1}{x-q} = \frac{1}{x-q^{r+1}} + \frac{q(1-q^r)}{(x-q^{r+1})(x-q^{r+2})} + \frac{q^2(1-q^r)(1-q^{r+1})}{(x-q^{r+1})(x-q^{r+2})(x-q^{r+3})} + \cdots.$$

Then we have

$$(10)\quad \frac{c_\nu}{(x-q)\cdots(x-q^\nu)} = \frac{c_\nu}{(x-q^2)(x-q^3)\cdots(x-q^{\nu+1})} + \frac{c_\nu q(1-q^\nu)}{(x-q^2)(x-q^3)\cdots(x-q^{\nu+2})} + \cdots .$$

The latter series converges absolutely and uniformly for all values of x such that $|x| \geqq r > |q|$. Let r_1 be the radius of absolute convergence of (1). Substituting from (10) into (1) we have a double series which we denote by $c_0 + \sum_{m,n=1}^{\infty} a_{mn}$. Let $b_{mn} = c_m q^{n-1}/x^{m+n-1}$. The series $\sum_{m,n=1}^{\infty} b_{mn}$ is the product of the two series $\sum_{\nu=1}^{\infty} c_\nu/x^\nu$ and $\sum_{\nu=1}^{\infty} q^{\nu-1}/x^\nu$, each of which converges absolutely if $|x| > R$, where R is the larger of the two numbers r and r_1. By a known theorem* $\sum_{m,n=1}^{\infty} b_{mn}$ converges absolutely in the same region. Since $\prod_{\nu=1}^{\infty}(1-q^\nu)$ and $\prod_{\nu=1}^{\infty}(1-q^{\nu+1}/x)$, $|x| > R$, represent convergent infinite products we can easily show that the ratio $|a_{mn}/b_{mn}| < M$, where M is finite. Hence $\sum_{m,n=1}^{\infty} |a_{mn}| < M \sum_{m,n=1}^{\infty} b_{mn}$ and the double series $\sum_{m,n=1}^{\infty} a_{mn}$ converges absolutely at the points of absolute convergence of $\sum_{m,n=1}^{\infty} b_{mn}$, that is, outside the circle of radius R. By a well known theorem,† the double series obtained above gives the same sum for any mode of summation. Summing by diagonals gives

$$(11)\quad c_0 + \sum_{\nu=1}^{\infty} \frac{S_\nu^{(1)}}{(x-q^2)(x-q^3)\cdots(x-q^{\nu+1})}, \qquad |x| > R,$$

where

$$S_\nu^{(1)} = \sum_{i=1}^{\nu} q^{\nu-i}(1-q^i)(1-q^{i+1})\cdots(1-q^{\nu-1})c_i.$$

In the same way we can show that the series (11) can be transformed into the series

$$c_0 + \sum_{i=1}^{\infty} \frac{S_\nu^{(2)}}{(x-q^3)(x-q^4)\cdots(x-q^{\nu+2})},$$

where

$$S_\nu^{(2)} = \sum_{i=1}^{\nu} q^{2(\nu-i)}(1-q^i)(1-q^{i+1})\cdots(1-q^{\nu-1})S_i^{(1)}$$

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* See Bromwich, *Theory of Infinite Series*, p. 90.

† Bromwich, *Theory of Infinite Series*, §31.

and finally into

$$c_0 + \sum_{\nu=1}^{\infty} \frac{S_\nu^{(k+1)}}{(x - q^{k+2})(x - q^{k+3}) \cdots (x - q^{k+\nu+1})},$$

where

$$S_\nu^{(k+1)} = \sum_{i=1}^{\nu} q^{(k+1)(\nu-i)}(1 - q^i)(1 - q^{i+1}) \cdots (1 - q^{\nu-1}) S_i^{(k)}.$$

Using this formula repeatedly we have

$$S_\nu^{(k)} = \sum q^{k(\nu-i)}(1 - q^i)(1 - q^{i+1}) \cdots (1 - q^{\nu-1}) q^{(k-1)(\nu-j)}(1 - q^j) \cdots (1 - q^{\nu-1}) \cdots q^{\nu-s}(1 - q^s)(1 - q^{s+1}) \cdots (1 - q^{\nu-1}) c_s,$$

where the summation is understood to be taken for all possible integral values of $\nu, i, j, \cdots, s$, such that

$$1 \leqq s \leqq \cdots \leqq j \leqq i \leqq \nu.$$

LEMMA.

$$\begin{aligned}(1 - q^{\alpha+1})&(1 - q^{\alpha+2}) \cdots (1 - q^{\alpha+k-1}) \\ &+ q^k(1 - q^\alpha)(1 - q^{\alpha+1}) \cdots (1 - q^{\alpha+k-2}) \\ &+ q^{2k}(1 - q^{\alpha-1})(1 - q^\alpha) \cdots (1 - q^{\alpha+k-3}) + \cdots \\ &+ q^{(\alpha-1)k}(1 - q^2)(1 - q^3) \cdots (1 - q^k) \\ &+ q^{\alpha k}(1 - q)(1 - q^2) \cdots (1 - q^{k-1}) \\ &= (1 - q^{\alpha+1})(1 - q^{\alpha+2}) \cdots (1 - q^{\alpha+k})/(1 - q^k).\end{aligned}$$

By combining terms in the first member two at a time, at each step taking the last two terms and continuing backward until all the terms are combined, we have the second member.

By simple induction we can establish the formulas

$$S_\nu^{(2)} = \sum_{i=0}^{\nu-1} \frac{q^i(1 - q^{i+1})(1 - q^{\nu-i})(1 - q^{\nu-i+1}) \cdots (1 - q^{\nu-1}) c_{\nu-i}}{1-q},$$

$$S_\nu^{(3)} = \sum_{i=0}^{\nu-1} \frac{q^i(1-q^{i+1})(1-q^{i+2})(1-q^{\nu-i})(1-q^{\nu-i+1}) \cdots (1-q^{\nu-1}) c_{\nu-i}}{(1 - q)(1 - q^2)},$$

and finally, by using the lemma above, we have by induction

$$S_\nu^{(k)} = \sum_{i=0}^{\nu-1} \frac{q^i(1 - q^{i+1}) \cdots (1 - q^{i+k-1})(1 - q^{\nu-i}) \cdots (1 - q^{\nu-1}) c_{\nu-i}}{(1 - q)(1 - q^2) \cdots (1 - q^{k-1})}.$$

Consider the product

$$\left[c_0 + \sum_{r=1}^{\infty} \frac{c_r}{(x-q)\cdots(x-q^r)}\right]\left[d_0 + \sum_{r=1}^{\infty} \frac{d_r}{(x-q)\cdots(x-q^r)}\right]$$

$$= c_0 d_0 + \sum_{r=1}^{\infty} \frac{c_r d_0 + c_0 d_r}{(x-q)\cdots(x-q^r)} + \left[\sum_{r=1}^{\infty} \frac{c_r}{(x-q)\cdots(x-q^r)}\right]\left[\sum_{r=1}^{\infty} \frac{d_r}{(x-q)\cdots(x-q^r)}\right].$$

Since we have

$$\sum_{r=1}^{\infty} \frac{c_r}{(x-q)\cdots(x-q^r)} = \sum_{r=1}^{\infty} \frac{S_r^{(k)}}{(x-q^{k+1})(x-q^{k+2})\cdots(x-q^{k+r})},$$

the above product becomes

$$c_0 d_0 + \sum_{r=1}^{\infty} \frac{A_r}{(x-q)(x-q^2)\cdots(x-q^r)},$$

where*

$$A_r = d_0 c_r + d_r c_0 + \sum_{i=1}^{r-1} S_i^{(r-i)}(c) d_{r-i}.$$

Within the region of uniform convergence of (1) the terms are all analytic functions and hence by a well known theorem† the series (1) can be differentiated term by term. If in (5) we write q^t for z, q^{r+1} for q, and denote the last term in the second member by L_k', then $\lim_{k\to\infty} L_k' = 0$ and we can write

$$\frac{1}{x-q^t} = \sum_{i=1}^{\infty} \frac{q^{(i-1)t}(1-q^{r-t+1})(1-q^{r-t+2})\cdots(1-q^{r-t+i-1})}{(x-q^{r+1})(x-q^{r+2})\cdots(x-q^{r+i})}, \qquad x \geqq r > |q|.$$

Using identities of this form to transform the series in the second member below into a geometrical factorial series, we have

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* $S_i^{(k)}(c)$ is written to indicate that the S's are formed with the c's. It will be observed that A_r is symmetric in the c's and d's.

† See for example Townsend, *Functions of a Complex Variable*, p. 224.

$$\frac{d}{dx}\left\{c_0 + \sum_{\nu=1}^{\infty} \frac{c_\nu}{(x-q)(x-q^2)\cdots(x-q^\nu)}\right\}$$

$$= -\sum_{\nu=1}^{\infty} \frac{c_\nu}{(x-q)(x-q^2)\cdots(x-q^\nu)}\left\{\frac{1}{x-q}+\frac{1}{x-q^2}+\cdots+\frac{1}{x-q^\nu}\right\}$$

$$= -\sum_{\nu=1}^{\infty}\sum_{j=1}^{\infty} \frac{C_\nu^{(j)}}{(x-q)(x-q^2)\cdots(x-q^{\nu+j})}$$

$$= \sum_{\nu=1}^{\infty} \frac{D_\nu}{(x-q)(x-q^2)\cdots(x-q^\nu)}, \quad |x| \geqq R,$$

where*

$$C_\nu^{(k)} = \sum_{i=0}^{\nu-1} \prod_{j=0}^{k-2} q^{i+1}(1-q^{\nu-i+j})c_\nu, \qquad D_\nu = -\sum_{j=1}^{\nu-1} C_{\nu-j}^{(j)},$$

and where R is defined above.

Again we can take the indefinite integral of the series (1) within the region of uniform convergence and obtain

$$\int\left\{c_0 + \sum_{\nu=1}^{\infty} \frac{c_\nu}{(x-q)(x-q^2)\cdots(x-q^\nu)}\right\}$$

$$= a_0 + a_1'x + a_0'' \log(x-q) + \sum_{\nu=1}^{\infty} \frac{a_\nu}{(x-q)(x-q^2)\cdots(x-q^\nu)},$$

where a_0 is the constant of integration,

$$a_0' = c_0, \quad a_0'' = c_1, \quad a_1 = -c_2,$$

and in general a_ν is given by the relation

$$a_\nu = -\frac{1}{\nu}\left\{c_{\nu+1} + \sum_{t=1}^{\nu} \sum_{i=1}^{\nu-t} \prod_{j=1}^{t} q^{it}(1-q^{\nu-t-i+j})a_{\nu-t}\right\}, \quad \nu > 1.$$

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* We interpret the notation so that $C_\nu' = \nu c_\nu$ and $D_1 = 0$.

NOTE ON FUNCTIONAL FORMS QUADRATIC IN A FUNCTION AND ITS FIRST p DERIVATIVES*

BY R. S. MARTIN

It has been shown that a normal functional form quadratic in a function and its derivative is reducible to an invariantive form quadratic in a function and a constant, and that the continuity of the coefficients of the first form implies the continuity of those of the second.† The result is here generalized to a form quadratic in a function and p derivatives.

Let y_0^α denote a continuous function of the real variable α defined on the range $a \leqq \alpha \leqq b$. Let $y_1^\alpha, y_2^\alpha, \cdots, y_p^\alpha$ denote the corresponding derivatives defined and continuous on the same interval. The normal quadratic form is

$$(A) \qquad Q = \sum_{i,j=0}^{p} A_{\alpha\beta}^{ij}\, y_i^\alpha y_j^\beta + A_\alpha^{ij}\, y_i^\alpha y_j^\alpha,$$

where $A_\alpha{}^i{}_\beta{}^j$ and $A_\alpha{}^{ij}$ are functions integrable in the Riemann sense, and where a Greek index repeated as a subscript and as a superscript is understood to denote integration with respect to that variable over the fundamental interval (a,b). The form Q is such that no generality is lost by assuming that $A_\alpha{}^i{}_\beta{}^j = A_\beta{}^j{}_\alpha{}^i$ and $A_\alpha{}^{ij} = A_\alpha{}^{ji}$. This assumption is made.

The convention of denoting integration by repeated Greek indices will be used in all that follows. In addition, when a Latin index (except p) occurs in a single term once as a superscript and once as a subscript, we shall understand summation with respect to that index over an integer range $(0, p-1)$. Thus

$$(A') \qquad \begin{aligned} Q = {} & A_{\alpha\beta}^{ij}\, y_i^\alpha y_j^\beta + A_{\alpha\beta}^{pj}\, y_p^\alpha y_j^\beta + A_{\alpha\beta}^{ip}\, y_i^\alpha y_p^\beta + A_{\alpha\beta}^{pp}\, y_p^\alpha y_p^\beta \\ & + A_\alpha^{ij}\, y_i^\alpha y_j^\alpha + A_\alpha^{pj}\, y_p^\alpha y_j^\alpha + A_\alpha^{ip}\, y_i^\alpha y_p^\alpha + A_\alpha^{pp} (y_p^\alpha)^2. \end{aligned}$$

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* Presented to the Society, December 30, 1930.

† A. D. Michal and L. S. Kennison, *Quadratic functional forms on a composite range*, Proceedings of the National Academy of Sciences, vol. 16 (1930), pp. 617–619.

We now state and prove the following theorems:

THEOREM 1. *The form Q may be written as a form quadratic in the single function $w^\alpha = y_p^\alpha$ and the p constants $y^i = y_i^a$, $(i=0, 1, \cdots, p-1)$.*

The reduction is readily accomplished by use of a remainder form of Taylor's theorem:

$$(1) \qquad y_i^\alpha = y_i^a + (\alpha - a)y_{i+1}^a + \cdots + \frac{(\alpha - a)^{p-i-1}}{(p-i-1)!} y_{p-1}^a + \int_a^\alpha \frac{(\alpha - \sigma)^{p-i-1}}{(p-i-1)!} y_p^\sigma d\sigma.$$

However, it is convenient to condense this by defining the functions

$$E_i^\alpha{}_\sigma = \frac{(\alpha - \sigma)^{p-i-1}}{(p-i-1)!}, \qquad \text{if } \alpha > \sigma,$$

$$= \frac{1}{2} \frac{(\alpha - \sigma)^{p-i-1}}{(p-i-1)!}, \text{ if } \alpha = \sigma,$$

$$= 0, \qquad \text{if } \alpha < \sigma;$$

$$E_i^\alpha{}_k = \frac{(\alpha - a)^{k-i}}{(k-i)!}, \qquad \text{if } k \geqq i,$$

$$= 0, \qquad \text{if } k < i.$$

The equation (1) then becomes simply

$$(2) \qquad y_i^\alpha = E_i^\alpha{}_k y^k + E_i^\alpha{}_\sigma w^\sigma.$$

If the latter result now be placed in the expression (A′) for Q, we have

$$(B) \qquad Q = B_{kl} y^k y^l + 2B_{k\sigma} y^k w^\sigma + B_{\sigma\tau} w^\sigma w_\tau + B_\sigma (w^\sigma)^2,$$

where

$$
\begin{aligned}
B_{kl} &= A_{\alpha\beta}^{ij} E_i^\alpha E_j^\beta + A_\alpha^{ij} E_i^\alpha E_j^\alpha, \\
B_{k\sigma} &= A_{\alpha\beta}^{ij} E_i^\alpha E_j^\beta + A_\alpha^{ij} E_i^\alpha E_j^\alpha + A_{\alpha\sigma}^{ip} E_i^\alpha + A_\sigma^{ip} E_k^{(\sigma)}, \\
B_{\sigma\tau} &= A_{\alpha\beta}^{ij} E_i^\alpha E_j^\beta + A_\alpha^{ij} E_i^\alpha E_j^\alpha + 2A_{\alpha\sigma}^{ip} E_i^\alpha + A_\tau^{pj} E_j^{(\tau)} + A_\sigma^{ip} E_i^{(\sigma)} \\
&\quad + A_{\sigma\tau}^{pp}, \\
B_\sigma &= A_\sigma^{pp},
\end{aligned}
\tag{3}
$$

where the indices in parentheses indicating suspension of the integration until the coefficient B has been associated with the function w. Thus we demonstrate the theorem by giving an explicit expression for the new coefficients B in terms of the coefficients A.

THEOREM 2. *If the coefficients A are continuous, the coefficients B as given by* (3) *are also continuous.*

This may be seen at once from the expressions (3). The coefficients B are the sums of certain of the coefficients A and a finite number of definite integrals of the products of the coefficients A by the functions E. Since the functions E are all continuous, except ${}_{p-1}E_\sigma^\alpha$, any discontinuities of B must arise in terms containing ${}_{p-1}E_\sigma^\alpha$. Consider then these terms. Terms of the type $A_\alpha {}^{ip-1}{}_i E_{kp-1}^\alpha E_\sigma^\alpha$ in which ${}_{p-1}E_\sigma^\alpha$ appears with one of its variables in integrated form must be continuous, for they are simply integrals in which the non-integrated variable of ${}_{p-1}E_\sigma^\alpha$ is one of the limits. For example,

$$
A_\alpha^{ip-1} {}_i E_k^\alpha \quad {}_{p-1}E_\sigma^\alpha = \int_\sigma^b (A_\alpha^{ip-1} {}_i E_k^{(\alpha)}) d\alpha.
$$

There is a possible exception in the term

$$
A_\alpha {}^{p-1\,p-1} {}_{p-1}E_\sigma^\alpha \; {}_{p-1}E_\sigma^\alpha,
$$

but we see that

$$
A_\alpha^{p-1\,p-1} {}_{p-1}E_\sigma^\alpha \quad {}_{p-1}E_\tau^\alpha = \int_{\mu(\sigma,\tau)}^b (A_\alpha^{p-1\,p-1}) d\alpha,
$$

where μ is equal to the greater of σ and τ. Being a continuous function of a continuous function, the term is continuous. The only terms in which ${}_{p-1}E_\sigma^a$ appears unintegrated are the terms $A_\sigma{}^{p-1p}{}_{p-1}E_\gamma{}^{(\sigma)}$ and $A{}^{p}{}_{\tau}{}^{p-1}{}_{p-1}E_\sigma{}^{(\gamma)}$. However, their sum is equal to $A_\mu{}^{p-1p}$ where μ is defined as above. Thus all the coefficients B are continuous,if the coefficients A are.

It has been shown elsewhere* that the form (B) of Q is invariantive; that is, a necessary and sufficient condition that Q vanish for all continuous functions w^a for which $w^a = w^b = 0$ and for all constants y^k is that the coefficients B be zero. Thus we may state

THEOREM 3. *A necessary and sufficient condition that a form quadratic in a function and its first p derivatives vanish for all functions $y_0{}^a$ whose pth derivative is continuous and vanishes at the end points is that the coefficients A satisfy the equations obtained by equating to zero the right members of the equations* (3).

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* See A. D. Michal, American Journal of Mathematics, 1928; A. D. Michal and T. S. Peterson, forthcoming issue of the Annals of Mathematics; A. D. Michal, Proceedings of National Academy, 1930; A. D. Michal, forthcoming issue of American Journal of Mathematics; A. D. Michal and L. S. Kennison, loc. cit.

PROPERTIES OF THE OPERATOR $z^{-\nu} \log z$, WHERE $z = d/dx$*

BY H. T. DAVIS

1. *Introduction.* A considerable bibliography with modern increments has been built up around the interpretation and use of the operators z^{ν} and $z^{-\nu}$, $z = d/dx$, where ν may assume fractional as well as integral values.† The peculiar efficacy of these symbols in the solution of integral equations of Volterra type on the one hand and in the resolution of difficulties in certain types of electrical transmission problems on the other has made a study of their properties of more than passing interest.

The essential peculiarity of the operators z^{ν} and $z^{-\nu}$ regarded as analytic functions is found in the fact that they possess branch points at the origin. It becomes of interest, therefore, to inquire into the existence of other operators with branch points at $z = 0$, as for example $\log z$. Wiener, in a paper which employs the Fourier transform of a function as its basis of definition, has established a rigorous foundation for the discussion of such branch point operators.‡ He illustrates his method by applying it to the operator $z^{1/2}$, but fails to give explicit consideration to $\log z$. In a subsequent paper F. Sbrana§ has supplied this deficiency by means of a method similar to that of Wiener in its use of the Fourier transform.

The original suggestion, however, is due to V. Volterra¶ who formulated it in a theory of *logarithms of composition*, which he applied effectively in the solution of the integral equation

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* Presented to the Society, April 3, 1931.

† See for example, J. D. Tamarkin, *On integrable solutions of Abel's integral equation*, Annals of Mathematics, vol. 31 (1930), pp. 218–229.

‡ *The operational calculus*, Mathematische Annalen, vol. 95 (1925–26), pp. 557–584.

§ *Sull'operazione infinitesimale nel gruppo delle derivazione*, Atti dei Lincei, (6), vol. 11 (1930), pp. 364–368.

¶ See V. Volterra and J. Pérès, *Leçons sur la Composition et les Fonctions Permutables*, Paris, 1924, Chap. 8. See also H. T. Davis, *The Inversion of Integrals of Volterra Type*, Indiana University, 1927, pp. 62–66.

$$f(x) = \int_0^x [\log (x - t) + C]u(t)dt, \quad f(0) = 0,$$

where $C = \lim_{\nu=1} -\Gamma'(\nu)/\Gamma(\nu) = 0.5772157 \cdots$ (Euler's constant).

It is some consequences of this suggestion, cast into the notation of derivatives, which we shall employ in this note. The present investigation will be limited to the formal aspects of the problem since we shall consider only what might be termed the *operational-theoretic* properties of the symbols.

2. *The Definition of* $z^{-\nu}$ log z *and its Inverse.* As is well known, generalized integration is defined by means of the formula

$$z^{-\nu} \rightarrow u(x) = \int_c^x \{(x - t)^{\nu-1}/\Gamma(\nu)\}u(t)dt,$$

where the symbol $\rightarrow$ is used to denote "operating upon." This becomes the definition of Riemann for $c = 0$ and the definition of Liouville for $c = -\infty$. Adopting the former for our present purposes, we replace $u(t)$ by $e^{(t-x)z} \rightarrow u(x)$ and thus attain the operational identity

$$z^{-\nu} = \int_0^x \{(x - t)^{\nu-1}e^{(t-x)z}/\Gamma(\nu)\}dt = \int_0^x \{s^{\nu-1}e^{-sz}/\Gamma(\nu)\}ds.$$

Let us now regard both terms of this equation as functions of the *continuous variable* ν and differentiate with respect to it. We thus obtain

$$(1) \quad -dz^{-\nu}/d\nu = z^{-\nu} \log z = \int_0^x [s^{\nu-1}e^{-sz}\{\psi(\nu) - \log s\}/\Gamma(\nu)]ds,$$

where $\psi(\nu) = \Gamma'(\nu)/\Gamma(\nu)$.

The formal reciprocal (the inverse operator) of this function is obtained by integrating $z^{-\mu+\nu}$ with respect to μ from 0 to ∞. We thus find

$$(2) \quad z^{\nu+1} \int_0^\infty z^{-\mu-1}d\mu = z^\nu/\log z = z^{\nu+1} \int_0^\infty \lambda(s)e^{-sz}ds,$$

where we use the abbreviation

$$(3) \quad \lambda(s) = \int_0^\infty \{s^\mu/\Gamma(\mu + 1)\}d\mu.$$

It is easily proved that the function $\lambda(s)$ is asymptotic to e^s in the positive interval. To see this we apply the Maclaurin integral test* for convergence to the series $e^s = 1+s+s^2/2! +s^3/3!+ \cdots$, and thus obtain the inequality

$$e^s - 1 \leqq \lambda(s) \leqq e^s.$$

Dividing by e^s, we have

$$1 - e^{-s} \leqq e^{-s}\lambda(s) \leqq 1,$$

which, for large positive values of s, establishes the desired property.

3. *Generalizations.* The definitions of the preceding section may be generalized in a useful way as follows. Let us denote by p the derivative operator $p=d/dv$, and by $\phi(p)$ a power series in p. We may then write

$$\phi(p) \to z^{-v} = \phi(p) \to \int_0^x \{(x-t)^{v-1}/\Gamma(v)\} e^{(t-x)z} dt,$$

from which we derive

$$(4) \quad z^{-v}\phi(-\log z) = \int_0^x \{\phi(p) \to (x-t)^{v-1}/\Gamma(v)\} e^{(t-x)z} dt.$$

For example, if $\phi(p)=p^2$, we obtain from (4) the formula

$$(5) \quad \begin{aligned} z^{-v}\log^2 z &= \int_0^x s^{v-1}e^{-sz}\{\log^2 s - 2\psi(v)\log s + \psi^2(v) - \psi'(v)\} ds/\Gamma(v), \\ &= \psi(v)z^{-v}\log z + \int_0^x s^{v-1}e^{-sz}\{\log^2 s - \psi'(v)\} ds/\Gamma(v). \end{aligned}$$

Similarly for the generalization of formula (2) we multiply $z^{-\mu+v}$ by $\theta(\mu)$ and integrate from 0 to ∞. We thus obtain

$$(6) \quad z^{v+1}\int_0^\infty z^{-\mu-1}\theta(\mu)d\mu = z^{v+1}\int_0^x e^{(t-x)z} I(x-t)dt,$$

where we abbreviate by means of the formula

$$I(s) = \int_0^\infty \{\theta(\mu)s^\mu/\Gamma(\mu+1)\} d\mu.$$

* T. J. Bromwich, *Infinite Series*, 2d ed. (1926), p. 33.

An example useful to us later is derived from the function $\theta(\mu)=e^{a\mu}$. We are thus able to derive

$$z^{\nu+1}\int_0^\infty z^{-\mu-1}\theta(\mu)d\mu = z^\nu\int_0^\infty e^{(a-\log z)\mu}d\mu = z^\nu/(a-\log z)$$

$$= z^{\nu+1}\int_0^x e^{(t-x)z}I(x-t)dt \tag{7}$$

where we write $I(s)=\int_0^\infty\{e^{a\mu}s^\mu/\Gamma(\mu+1)\}d\mu$.

4. *Properties of the Operators.* In the subsequent use of the operators defined in the preceding sections it will be important to know the order in which they may be applied. This question is answered in the following theorems.

THEOREM 1.

$$z^{-\mu}\rightarrow(z^{-\nu}\log z) = z^{-\mu-\nu}\log z. \tag{8}$$

PROOF. By definition we have

$$z^{-\mu}\rightarrow(z^{-\nu}\log z) = \int_0^x\{(x-t)^{\mu-1}/\Gamma(\mu)\}dt\int_0^t\{\psi(\nu)$$

$$-\log(t-s)\}(t-s)^{\nu-1}e^{(s-x)z}ds,$$

$$=\int_0^x e^{(s-x)z}ds\int_s^x\{\psi(\nu)$$

$$-\log(t-s)\}(x-t)^{\mu-1}(t-s)^{\nu-1}dt/[\Gamma(\mu)\Gamma(\nu)].$$

Making the transformation $y=(t-s)/(x-s)$, we can write this in the form

$$z^{-\mu}\rightarrow(z^{-\nu}\log z) = \int_0^x e^{(s-x)z}J(x-s)ds/[\Gamma(\mu)\Gamma(\nu)], \tag{9}$$

where we have

$$J(x-s) = (x-s)^{\mu+\nu-1}\{\psi(\nu)\Gamma(\mu)\Gamma(\nu)/\Gamma(\mu+\nu)$$

$$-\int_0^1\log[(x-s)y](1-y)^{\mu-1}y^{\nu-1}dy\}.$$

From the identity

$$\int_0^1 \log y(1-y)^{\mu-1}y^{\nu-1}dy = \{\psi(\nu) - \psi(\mu+\nu)\}B(\mu,\nu),^*$$

where $B(\mu, \nu)$ is the Eulerian integral of first kind, we can then write

$$J(x-s) = (x-s)^{\mu+\nu-1}B(\mu,\nu)\{\psi(\mu+\nu) - \log(x-s)\}.$$

When this function is replaced in (9) we see that the integral reduces to the definition of $z^{-\mu-\nu}\log z$ which establishes the theorem.

THEOREM 2.

$$z^{-\nu}\log z \to z^{-\mu} = z^{-\mu} \to z^{-\nu}\log z = z^{-\mu-\nu}\log z. \tag{10}$$

PROOF. Giving our attention to the left member we have, by definition,

$$z^{-\nu}\log z \to z^{-\mu} = z^{-\nu}\log z \to \int_0^x (x-s)^{\mu-1}e^{(s-x)z}ds/\Gamma(\mu)$$

$$= \int_0^z (x-t)^{\nu-1}\{\psi(\nu) - \log(x-t)\}dt$$

$$\cdot \int_0^t (t-s)^{\mu-1}e^{(s-x)z}ds/[\Gamma(\mu)\Gamma(\nu)]$$

$$= \int_0^z e^{(s-x)z}ds$$

$$\cdot \int_s^z \{\psi(\nu) - \log(x-t)\}(x-t)^{\nu-1}(t-s)^{\mu-1}dt/[\Gamma(\mu)\Gamma(\nu)].$$

Making the transformation $y=(t-s)/(x-s)$, we can then write

$$z^{-\nu}\log z \to z^{-\mu} = \int_0^z e^{(s-x)z}K(x-s)ds/[\Gamma(\mu)\Gamma(\nu)], \tag{11}$$

where we abbreviate as follows:

$$K(x-s) = (x-s)^{\nu+\mu-1}\int_0^1 \{\psi(\nu)(1-y)^{\nu-1}y^{\mu-1}$$

* See N. Nielsen, *Handbuch der Gammafunktion*, Leipzig, 1906, p. 172.

$$-\log(x-s)(1-y)^{\nu-1}y^{\mu-1}-\log(1-y)(1-y)^{\nu-1}y^{\mu-1}\Big\}dy$$
$$= (x-s)^{\nu+\mu-1}\Gamma(\mu)\Gamma(\nu)\left[\psi(\mu+\nu)-\log(x-s)\right].$$

When this is substituted in (11), the integral becomes the definition of $z^{-\mu-\nu}\log z$. But, by Theorem 1, this is equivalent to $z^{-\mu}\to z^{-\nu}\log z$, which completes the proof.

THEOREM 3.

$$z^{-\nu}\log z \to z^{-\mu}\log z = z^{-\mu-\nu}\log^2 z. \tag{12}$$

PROOF. We have, by definition,

$$z^{-\nu}\log z \to z^{-\mu}\log z$$
$$= \int_0^x (x-t)^{\nu-1}\left[\psi(\nu)-\log(x-t)\right]$$
$$\cdot \int_0^t (t-s)^{\mu-1}\left[\psi(\mu)-\log(t-s)\right]e^{(s-x)z}ds,$$
$$= \int_0^x e^{(s-x)z}ds$$
$$\cdot \int_s^x (x-t)^{\nu-1}(t-s)^{\mu-1}\left[\psi(\nu)-\log(x-t)\right]\left[\psi(\mu)-\log(t-s)\right]dt.$$

Employing the transformation $y=(t-s)/(x-s)$, we find

$$z^{-\nu}\log z \to z^{-\mu}\log z = \int_0^x e^{(s-x)z}L(x-s)ds,$$

where we abbreviate thus:

$$L(x-s) = (x-s)^{\mu+\nu-1}\int_0^1 y^{\mu-1}(1-y)^{\nu-1}\left[\psi(\nu)-\log(x-s)\right.$$
$$\left.-\log(1-y)\right]\left[\psi(\mu)-\log(x-s)-\log y\right]dy. \tag{13}$$

Computing the partial derivatives $\partial B(\mu,\nu)/\partial\mu$, $\partial B/\partial\nu$, and $\partial^2 B/\partial\mu\partial\nu$, where $B(\mu,\nu)=\int_0^1 y^{\mu-1}(1-y)^{\nu-1}dy$, we get the integrals

$$(14)\quad \begin{cases} \displaystyle\int_0^1 \log y(1-y)^{\nu-1}y^{\mu-1}dy = \{\psi(\nu) - \psi(\mu+\nu)\}B(\mu,\nu), \\ \displaystyle\int_0^1 \log (1-y)(1-y)^{\nu-1}y^{\mu-1}dy = \{\psi(\nu) - \psi(\mu+\nu)\}B(\mu,\nu), \\ \displaystyle\int_0^1 \log y \log (1-y)(1-y)^{\nu-1}y^{\mu-1}dy = \{\psi(\mu)\psi(\nu) \\ \quad - [\psi(\mu)+\psi(\nu)]\psi(\mu+\nu) + \psi^2(\mu+\nu) \\ \quad - \psi'(\mu+\nu)\}B(\mu,\nu). \end{cases}$$

When the integrand of (13) is expanded and the values of (14) substituted, we get

$$L(x-s) = (x-s)^{\mu+\nu-1}B(\mu,\nu)[\log^2(x-s) - 2\psi(\mu+\nu)\log(x-s) + \psi^2(\mu+\nu) - \psi'(\mu+\nu)].$$

From this, taking note of (5), we derive the proof of the theorem.

THEOREM 4.

$$z^{-\nu}\log^n z \to z^{-\mu}\log^m z = z^{-\mu-\nu}\log^{n+m} z,$$

where n and m are positive integers.

PROOF. In the proof of this theorem, which is seen to include the preceding as a special case, we shall employ a slightly different argument. We have, by definition,

$$z^{-\nu}\log^n z \to z^{-\mu}\log^m z$$

$$= (-1)^{n+m}\frac{\partial^n}{\partial\nu^n}\int_0^x \{(x-t)^{\nu-1}/\Gamma(\nu)\}dt \cdot \frac{\partial^m}{\partial\mu^m}\int_0^t \{(t-s)^{\mu-1}/\Gamma(\mu)\}e^{(s-x)z}ds,$$

$$= (-1)^{n+m}\int_0^x e^{(s-x)z}ds \cdot \int_s^x \frac{\partial^{n+m}}{\partial\nu^n\partial\mu^m}\{(x-t)^{\nu-1}(t-s)^{\mu-1}/[\Gamma(\mu)\Gamma(\nu)]\}dt.$$

Making the customary transformation, we find

$$z^{-\nu}\log^n z \to z^{-\mu}\log^m z$$

$$= (-1)^{n+m}\int_0^x e^{(s-x)z}ds \cdot \int_0^1 \frac{\partial^{n+m}}{\partial\nu^n\partial\mu^m}\{(x-s)^{\mu+\nu-1}(1-y)^{\nu-1}y^{\mu-1}/[\Gamma(\mu)\Gamma(\nu)]\}dt$$

$$= (-1)^{n+m}\frac{\partial^{n+m}}{\partial\nu^n\partial\mu^m}\int_0^x e^{(s-x)z}\{(x-s)^{\mu+\nu-1}/\Gamma(\mu+\nu)\}ds$$

$$= z^{-\nu-\mu}\log^{n+m} z,$$

which was to be proved.

5. *The Rule of Leibnitz.* In the application of operators the following generalization of the formula of Leibnitz for the nth derivative of a product is often of great importance:

$$(15)\quad F(x,z)\to uv = vF(x,z)\to u + v'F_z'(x,z)\to u/1! + v''F_z''(x,z)\to u/2! + \cdots,$$

where we employ the abbreviation $F_z^{(n)}(x,z)=\partial^n F(x,z)/\partial z^n$.*

Although it has been proved that this formula applies when $F(x,z)\equiv F(z)$ is a rational function of z, existing discussions of its validity do not include operators of the form $z^{-\nu}\log z$. It therefore becomes of interest to prove the following theorem.

THEOREM 5. *The generalized Leibnitz formula* (15) *applies when* $F(x,z)=z^{-\nu}\log z$.

PROOF. By definition, we have

$$z^{-\nu}\log z\to uv = \int_0^x (x-t)^{\mu-1}\{\psi(\nu)-\log(x-t)\}u(t)v(t)dt/\Gamma(\nu)$$

$$= \int_0^x \{\psi(\nu)-\log(x-t)\}u(t)\cdot\left\{\sum_{n=0}^{\infty} v^{(n)}(x)(x-t)^{\nu+n-1}(-1)^n/n!\right\}dt/\Gamma(\nu).$$

* This formula was originally due to C. J. Hargreave, London Philosophical Transactions, vol. 138 (1848), p. 31. See also S. Pincherle, *Opérations fonctionnelles*, Encyclopédie des Sciences Mathématiques, part 2, vol. 4, No. 26, in particular p. 10.

But since $\psi(\nu)=\psi(\nu+1)-1/\nu$, $\Gamma(\nu)=\Gamma(\nu+1)/\nu$, we can write

$$z^{-\nu}\log z \to u\,v = v(x)z^{-\nu}\log z \to u - \int_0^x \sum_{n=1}^{\infty} \Big\{\log(x-t) - \psi(\nu+n) + \frac{1}{\nu} + \frac{1}{(\nu+1)} + \cdots + \frac{1}{(\nu+n-1)}\Big\} v^{(n)}(x) \cdot (x-t)^{\nu+n-1}(-1)^n u(t)\,dt \frac{\nu(\nu+1)\cdots(\nu+n-1)}{\Gamma(\nu+n)n!}$$

$$= \Big[v(x)z^{-\nu}\log z + \sum_{n=1}^{\infty}(-1)^n v^{(n)}(x) \cdot \frac{\{\nu(\nu+1)\cdots(\nu+n-1)z^{-\nu-n}\log z - P_n(\nu)z^{-\nu-n-1}\}}{n!}\Big] \to u(x),$$

where we write

$$P_1(\nu) = 1,\ P_n(\nu) = (\nu+1)\cdots(\nu+n-1) + \nu(\nu+2)\cdots(\nu+n-1) + \nu(\nu+1)(\nu+3)\cdots(\nu+n-1) + \cdots + \nu(\nu+1)\cdots(\nu+n-2).$$

But the coefficient of $v^{(n)}(x)/n!$ is precisely the nth derivative of $z^{-\nu}\log z$, which was to be proved.

The following more general result follows as a consequence of the preceding theorem.

THEOREM 6. *The generalized Leibnitz formula applies when $F(x, z)=z^{-\nu}\log^n z$, where n is a positive integer.*

PROOF. We employ induction. In Theorem 5 we have shown that the theorem is true for $n=1$. Let us now assume that it also holds for $n=k$. Making the abbreviations $\phi(z)=z^{-\nu}\log^k z$ and $\rho(z)=\log z$, we shall then immediately obtain

$$\rho(z) \to (\phi \to uv) = \rho \to \{[v\phi + v'\phi'/1! + v''\phi''/2! + \cdots] \to u(x)\}.$$

But since we have shown above that the theorem is true for $\rho(z)$, we at once deduce

$$\rho(z) \to (\phi \to u\,v) = \{v\rho\phi + v'(\rho'\phi + \phi'\rho)/1! + v''(\rho''\phi + 2\rho'\phi' + \rho\phi'')/2! + \cdots\} \to u(x).$$

But, by Theorem 4, we have $\rho(z)\to\phi(z)=\rho\phi$. This fact combined with the preceding equation shows that the proposition

is also true for $n=k+1$. The proof is then completed by induction in the customary manner.

6. *The Bourlet Operational Product.* For the application of the operator $z^{-\rho}\log z$ to problems in the theory of functional equations it is necessary to know whether it belongs to the class of operators the symbolic multiplication of which is determined by the Bourlet product:

$$(16)\quad S(x,z)\rightarrow T(x,z)=[S\cdot T](x,z)=ST+(\partial T/\partial x)(\partial S/\partial z)/1!+(\partial^2T/\partial x^2)(\partial^2S/\partial z^2)/2!+\cdots.$$

The following theorem will now be proved.

THEOREM 7. *If $S(x,z)$ and $T(x,z)$ are operators the symbolic multiplication of which is given by the Bourlet product, then $z^{-\rho}\log^n z S(x,z)$ and $z^{-\mu}\log^m z\,T(x,z)$ are also such operators.*

PROOF. If we abbreviate $\phi(z)=z^{-\rho}\log^n z$ and $\rho(z)=z^{-\mu}\log^m z$, our problem is to show that

$$\phi(z)S\rightarrow\rho T=\rho[ST+(\partial T/\partial x)(\partial\phi S/\partial z)/1!+(\partial^2T/\partial x^2)(\partial^2\phi S/\partial z^2)/2!+\cdots].$$

To prove this we expand $S(x,z)$ and $T(x,z)$ in the series

$$S(x,z)=\sum_{n=-\infty}^{\infty}S_n(x)z^n,\quad\text{and}\quad T(x,z)=\sum_{n=-\infty}^{\infty}T_n(x)z^n.*$$

We can then write

$$A(x,z)\equiv(\phi S)\rightarrow\sum_{n=-\infty}^{\infty}\rho T_n(x)z^n=\sum_{n=-\infty}^{\infty}(\phi S)\rightarrow\rho T_n(x)z^n.$$

But since we have shown in Theorem 6 the validity of the Leibnitz rule for ϕz^n, in particular, and hence formally for $\sum_{n=-\infty}^{\infty}S_n(x)\phi z^n$, in general, we are able to write

$$\begin{aligned}A(x,z)&=\sum_{n=-\infty}^{\infty}\sum_{m=0}^{\infty}(d^mT_n/dx^m)(\partial^m\phi S/\partial z^m)\rho z^n/m!\\&=\sum_{m=0}^{\infty}(\partial^m\phi S/\partial z^m)\sum_{n=-\infty}^{\infty}(d^mT_n/dx^m)\rho z^n/m!\\&=\rho\sum_{m=0}^{\infty}(\partial^mS/\partial z^m)(\partial^mT/\partial x^m)/m!.\end{aligned}$$

We thus attain the desired equation and the theorem is proved.

* For the generality of these expansions, see C. Bourlet, Annales de l'École Normale Supérieure, (3), vol. 14 (1897), p. 150.

7. *Applications.* Three simple examples will now be given to show the natural way in which the logarithmic operator may be expected to appear in the solution of functional equations.

EXAMPLE 1. (Volterra's problem.) Solve the integral equation

$$f(x) = \int_0^x [\log (x - t) + A]u(t)dt, \qquad f(0) = 0.$$

Writing this equation symbolically, we have

$f(x) = (-z^{-1} \log z + az^{-1}) \rightarrow u(x)$, $a = A - C$ (Euler's constant),

and, by Theorem 7, we find

$$u(x) = z/(a - \log z) \rightarrow f(x).$$

The interpretation of this symbol follows at once from (7):

$$u(x) = z^2 \rightarrow \int_0^x f(t)I(x - t)dt,$$

where

$$I(s) = \int_0^\infty \{e^{a\mu}s^\mu/\Gamma(\mu + 1)\}d\mu.$$

EXAMPLE 2. (Volterra.) Solve the integral equation

$$f(x) = \int_0^x [\log^2 (x - t) + A \log (x - t) + B]u(t)dt.$$

This equation can be written symbolically in the form

$$f(x) = z^{-1}(\log^2 z + \alpha \log z + \beta) \rightarrow u(x),$$

where we abbreviate,

$$\alpha = -A - 2\psi(1), \quad \beta = B + A\psi(1) + \psi^2(1) + \psi'(1).$$

The solution then appears in the form

$$\begin{aligned} u(x) &= z/(\log^2 z + \alpha \log z + \beta) \rightarrow f(x) \\ &= z[\phi_1/(\log z - \lambda_1) + \phi_2/(\log z - \lambda_2)] \rightarrow f(x), \end{aligned}$$

where $\phi_1 = -\phi_2 = 1/(\lambda_1 - \lambda_2)$, and where λ_1 and λ_2 are roots of the equation $\lambda^2 + \alpha\lambda + \beta = 0$. The solution, when λ_1 and λ_2 are distinct, is thus seen to be attainable by means of the operator of Example 1.

When $\lambda_1=\lambda_2=\lambda$, the preceding solution is replaced by

$$u(x) = [z/(\log z - \lambda)^2] \rightarrow f(x)$$
$$= z^2 \rightarrow \int_0^x R(x-t)f(t)dt,$$

where

$$R(s) = \int_0^\infty \{s^\mu \mu e^{\lambda\mu}/\Gamma(\mu+1)\}d\mu.$$

EXAMPLE 3. Discuss the operator inverse to $X(D)=1-D$, $D=xz$. This operator is found to be $Y(D) = 1 - De^{-D}\mathrm{li}(e^D)$, where we abbreviate

$$\mathrm{li}(e^D) = \int_{-D}^\infty e^{-t}dt/t.$$

To verify this we compute the Bourlet product (16) and thus obtain

$$[X\cdot Y] = (1-D)[1 - De^{-D}\mathrm{li}(e^D)]$$
$$+ D[e^{-D}\mathrm{li}(e^D) - De^{-D}\mathrm{li}(e^D) + 1] \equiv 1.$$

Employing a well known expansion for the function $\mathrm{li}(e^D)$, we can write $Y(D)$ in the form,

$$Y(D) = 1 - De^{-D}[C + \log D + D + D^2/(2\cdot 2!) + D^3/(3\cdot 3!) + \cdots],$$

where C is Euler's constant.* If we note that

$$e^{-D}F(z) \rightarrow f(x) = e^{-D} \rightarrow [F(z) \rightarrow f(x)] = \lim_{z=0} [F(z) \rightarrow f(x)],$$

where $F(z)$ is a rational function of z, and from our definition that $e^{-D}\log z \rightarrow f(x)=k$ (a constant), we may then write

$$Y(D) \rightarrow f(x) = Ax + 1 - x\log x - \sum_{n=2}^\infty x^n f^{(n)}(0)/[(n-1)\cdot(n-1)!],$$

where A is a constant. It can be verified without difficulty that this is the operational form of the solution of the differential equation $u(x)-xu'(x)=f(x)$.

WATERMAN INSTITUTE, INDIANA UNIVERSITY

* See Bromwich, *Infinite Series*, 2d ed., 1926, p. 334.

WHITEHEAD AND RUSSELL'S THEORY OF DEDUCTION AS A MATHEMATICAL SCIENCE*

BY B. A. BERNSTEIN

1. *Introduction.* Whitehead and Russell develop in their *Principia Mathematica* a theory of deduction for "elementary" propositions.† On this theory almost the entire *Principia* rests. From the primitives (the undefined ideas and the postulates) of this theory and from a few other primitives the authors proceed to derive all logic and all mathematics. It is then of great importance to the mathematician and to the logician to understand clearly the mathematical make-up of the theory of deduction. But such understanding is not easy. The novel views of the *Principia*, its strange symbolism, its elaborate structure, its long informal discussions, all combine to hide the real mathematical nature of the theory from the average man interested in mathematics and in logic. The purpose of this paper is to examine the theory critically and to exhibit clearly its true nature as a mathematical science.

My discussion will be based on the formal, the "official" development of the theory as found in the first edition of the *Principia* and as modified by the second edition. The informal discussions will be ignored unless they furnish needed information not obtainable from the formal account.‡

I begin with a brief account of the theory.

2. *Brief Account of the Theory.* The theory of deduction is "the calculus of propositions" (p. 88); it is "the theory of how one proposition can be inferred from another" (p. 90). As a mathematical science, as "the most elementary part of mathematics" (p. 115), the theory consists of a certain set of primitive

* Presented to the Society, June 20, 1929, as part of a paper entitled *On Whitehead and Russell's theory of deduction.*

† Whitehead and Russell, *Principia Mathematica*, vol. 1, second edition, pp. 87–126. In later references to pages, this volume of the *Principia* will be understood.

‡ There are some discrepancies between the informal and the formal accounts of the theory.

propositions and of theorems derived from the primitive propositions. The authors prove neither the consistency nor the independence of their primitive propositions, because "the recognized methods of proving independence" (and impliedly also consistency) "are not applicable, without reserve, to fundamentals" (p. 91, footnote).

There are two editions of the *Principia*. In the first edition the primitive ideas are six in number, as follows:

(1) *Elementary proposition*, denoted by p, q, r, $\cdots$;

(2) *Elementary propositional function of* x, denoted by "ϕx";

(3) *Assertion* of an elementary proposition p, denoted by "$\vdash \cdot p$";

(4) [*Assertion of an elementary propositional function* ϕx, denoted by "$\vdash \cdot \phi x$"];

(5) *Negation* of an elementary proposition p, denoted by "$\sim p$";

(6) *Disjunction* of two elementary propositions p and q, denoted by "$p \vee q$".

In the second edition the authors tell us (p. xiii) to drop notion (4) from the list of primitives. To indicate this fact, I have placed this notion in brackets.

The authors explain (pp. 88–93) their primitive ideas. "Elementary propositions" are "such as contain no reference, explicit or implicit, to any totality" (p. 88). An "elementary propositional function ϕx" is an expression containing a variable x such that when x is determined, ϕx becomes an elementary proposition. The symbol "$\vdash$" means "it is true that," and $\vdash \cdot p$ is to be distinguished from plain p. The symbol "$\sim p$" our authors read: " 'not-p' or 'p is false'." The disjunction "$p \vee q$" they read: " 'p or q', that is, 'either p is true or q is true'."

Before stating the primitive propositions the authors introduce a definition of the "implicative function" $p \supset q$. The definition is:

$$*1{\cdot}01. \qquad (p \supset q) = (\sim p \vee q)\ Df.$$

The authors read this: " 'p implies q' is to be defined to mean 'either p is false or q is true'."

In the first edition the primitive propositions are ten in number. They are (with the original numbering retained and with

the Peano dot-parentheses replaced by ordinary parentheses) propositions *1·1—*1·72 below. In the second edition we are told to omit *1·11 from the list of primitive propositions. I have placed this proposition in brackets. Following each of the propositions *1·2—*1·6 is the authors' interpretation of the symbols. The primitive propositions follow.

*1·1. Anything implied by a true elementary proposition is true.

Our authors say "we cannot express this principle symbolically."

*1·11. [When ϕx can be asserted, where x is a real variable, and $\phi x \supset \psi x$ can be asserted, where x is a real variable, then ψx can be asserted, where x is a real variable.]

*1·2. $\vdash \cdot (p \vee p) \supset p.$

"If either p is true or p is true, then p is true."

*1·3. $\vdash \cdot q \supset (p \vee q).$

"If q is true, then 'p or q' is true."

*1·4. $\vdash \cdot (p \vee q) \supset (q \vee p).$

" 'p or q' implies 'q or p'."

*1·5. $\vdash \cdot \{p \vee (q \vee r)\} \supset \{q \vee (p \vee r)\}.$

"If either p is true, or 'q or r' is true, then either q is true or 'p or r' is true."

*1·6. $\vdash \cdot (q \supset r) \supset \{(p \vee q) \supset (p \vee r)\}.$

"If q implies r, then 'p or q' implies 'p or r'."

*1·7. If p is an elementary proposition, $\sim p$ is an elementary proposition.

"1.71. If p and q are elementary propositions, $p \vee q$ is an elementary proposition."

*1·72. If ϕp and ψp are elementary propositional functions which take elementary propositions as arguments, $\phi p \vee \psi p$ is an elementary propositional function.

In the first edition much is made of the distinction between "real" and "apparent" variables, but in the second edition we are instructed (p. xiii) to abandon this distinction and to read a proposition of the form "$\vdash \cdot \phi p$" as if it were written "$\vdash \cdot (p) \cdot \phi p$," that is, "$\phi p$ is true for all p's."

The authors define the product $p \cdot q$ of two propositions p and q by:

*3·01. $(p \cdot q) = \{\sim(\sim p \vee \sim q)\}\ Df.$

" 'p and q are both true' is defined to be 'it is false that either p is false or q is false'."

The *equivalence* of p and q, denoted by "$p \equiv q$," is defined by:

*4·01. $(p \equiv q) = (p \supset q)(q \supset p)\ Df.$

"p and q are said to be *equivalent* when p implies q and q implies p" (p. 7).

The theorems derived from the primitive propositions are all, with the single exception of *3·03,† of the form $\vdash \cdot \phi(p, q, r, \cdots)$, where $\phi(p, q, r, \cdots)$ is an elementary proposition built up from $p, q, r, \cdots$ by means of the operations "$\sim$" and "$\vee$," that is, the theorems are all, with the single exception of *3·03, of the type of propositions *1·2—*1·6. In proving a theorem of the form $\vdash \cdot \phi(p, q, r, \cdots)$ the authors restrict themselves to two general methods. One is to find a *known* proposition $\vdash \cdot f(p, q, r \cdots)$ of which $\vdash \cdot \phi(p, q, r, \cdots)$ is a particular case got from $\vdash \cdot f(p, q, r \cdots)$ by substituting particular forms of elementary propositions for the general propositions $p, q, r, \cdots$. The other method is to find a function $f(p, q, r, \cdots)$ such that we have both $\vdash \cdot f(p, q, r, \cdots)$ and $\vdash \cdot f(p, q, r, \cdots) \supset \phi(p, q, r, \cdots)$, and then apply *1·11. The demonstrations of the early theorems are given in full, except that reference to *1·7, *1·71, *1·72 is generally tacit. The demonstrations are given in symbols, the authors employing for this purpose the symbols of the theory and other symbols.

This concludes my report of the theory of deduction.‡ In this report I tried to give a mere account of the theory as presented by the authors, an account without comment. What now shall we say of the mathematical nature of the theory? To answer this question I shall first give a brief characterization of a mathematical science, a characterization which an analysis of any orthodox mathematical science seems to me to justify.

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† Proposition *3·03 states that given $\vdash \cdot \phi p$ and $\vdash \cdot \psi p$ then we have $\vdash \cdot \phi p \cdot \psi p$.

‡ In the second edition, the authors give a great deal of space to a restatement of their theory into a language in which Sheffer's operation "$|$" is made basic. I have given no account of this restatement in my report, because it is irrelevant to the discussion of the theory as a mathematical science.

3. *Nature of a Mathematical Science.* A mathematical science, a science in the sense of a pure deductive theory, is a body of propositions consisting of *postulates* and *theorems*. The postulates are the propositions which cannot be derived from one another; the theorems are the propositions which are derivable from the postulates. The propositions of a mathematical science concern a certain totality of things and certain connections among the things; they give information about a certain *class* of *elements* and about certain *operations* or *relations* among the elements. The classes, operations and relations constitute the *ideas* of the science. Some of the ideas are *primitive*, that is, not definable in terms of one another; the rest are definable in terms of the primitive ones. Since the propositions of the science give information *about* its ideas, every proposition must contain, beside the ideas *belonging to* the science, also ideas that are *outside* the science. The latter are the ideas of general language by means of which the information is given. Likewise, since the theorems are derived from the postulates, the science must use, beside the propositions belonging to it, also propositions which are outside it. These are the principles of logic which give the theorems as conclusions from the postulates as premises.†

A mathematical science, in the sense of a pure deductive theory, is *abstract*. If, for example, a *class* K, a *binary operation* $\oplus$, and a *dyadic relation* $\ominus$ constitute the primitive ideas of a mathematical science, then K, $\oplus$, $\ominus$ have no properties other than that K is some class of elements, $\oplus$ some binary operation, and $\ominus$ some dyadic relation. K is simply some set of things, denoted, say, by $a, b, c, \cdots$; $\oplus$ is merely some rule which states for every pair of K-elements a, b what K-element $a \oplus b$, if any, corresponds to a, b; $\ominus$ is merely a rule which states for every pair of K-elements a, b whether or not the proposition $a \ominus b$ is true. All other properties of K, $\oplus$, $\ominus$ are given by the postulates. The science is the *system* $(K, \oplus, \ominus)$ satisfying the postulates. It is the blank form, the logical skeleton, for a number of concrete sciences each of which is obtainable from $(K, \oplus, \ominus)$

† On the necessity for distinguishing between the ideas and propositions belonging to a science and the ideas and propositions that are outside it, compare my *Sets of postulates for the logic of propositions*, Transactions of this Society, vol. 28, p. 472, last footnote. See also my review of the *Principia*, this Bulletin, vol. 32, pp. 711–713.

by substituting for K, $\oplus$, $\ominus$ respectively a concrete class, a concrete operation, and a concrete relation so as to make the postulates true propositions. It is this abstract nature of a mathematical science that makes possible the ordinary methods of proving the consistency and the independence of its postulates: the postulates of the above system (K, $\oplus$, $\ominus$) are consistent if K, $\oplus$, $\ominus$ can be so interpreted as to make all the postulates true; a postulate P is independent of the other postulates if K, $\oplus$, $\ominus$ can be so interpreted as to contradict P and not contradict any of the rest.

How does the theory of deduction of the *Principia* meet the requirements of a mathematical science just outlined? A brief consideration of the primitives will show that the theory falls quite short of these requirements both with regard to the primitive ideas and with regard to the primitive positions.

4. *Shortcomings of the Theory as a Mathematical Science.* As to the primitive ideas of the theory, we observe that the ideas denoted by p, $\sim p$, $p \vee q$ fully conform to the demands made on the primitive ideas of a mathematical science: p is a member of the class of elementary propositions, the totality with which the theory is concerned; $\sim p$ and $p \vee q$ denote results of operations in the class of elementary propositions. But the ideas represented by "ϕx" and "$\vdash \cdot p$" do not satisfy the above demands. ϕx can be *defined* in terms of the primitive ideas p, $\sim p$, $p \vee q$. For ϕx is used in the theory simply as an expression involving an elementary proposition x built up from x by means of "$\sim$" and "$\vee$", and hence is an *elementary proposition*, by $*1 \cdot 7$ and $*1 \cdot 71$. The idea "$\vdash \cdot p$" does not conform to the above demands because it does not stand for any *class* or *operation* or *relation*. Moreover, nothing is said *about* it, no condition is imposed on it, by the primitive propositions.

As to the primitive propositions of the theory, we observe that $*1 \cdot 7$ and $*1 \cdot 71$ are the only propositions that meet our requirements. $*1 \cdot 7$ and $*1 \cdot 71$ are the only primitive propositions which contain both ideas of the theory, about which information is given, and also ideas outside the theory, which give the information. Proposition $*1 \cdot 1$ does not have the necessary requirements because it *has no symbols*, hence, impliedly, has no ideas belonging to the theory, hence says nothing about the ideas of the theory. Propositions $*1 \cdot 2 - *1 \cdot 6$ have not the requirements

because they *contain only symbols denoting ideas of the theory*, and hence do not say anything about these ideas. Finally, proposition *1·72 does not satisfy the requirement of a primitive proposition because *it can be derived as a theorem* by means of *1·7 and *1·71, since, as we have seen, ϕx and ψx are elementary propositions.

The presence of *1·1 — *1·6 among the primitive propositions removes the theory from the category of abstract sciences, and would account for the authors' view that the recognized methods of proving independence are not applicable to their theory.

To the above defects in the primitives, as found in the first edition and retained in the second, must be added one introduced in the second edition. In this edition we are told to read every proposition of the form $\vdash \cdot \phi(p, q, \cdots)$ as if it were written $\vdash \cdot (p, q, \cdots) \cdot \phi(p, q, \cdots)$. But we are nowhere told whether the symbol $(p, q, \cdots) \cdot \phi(p, q, \cdots)$ denotes a primitive idea or not. In the first edition this symbol stands for a primitive idea carefully relegated to the theory of "apparent" variables.

5. *The Theory Transformed into a Mathematical Science.* But serious as are the shortcomings of the theory as a mathematical science, its primitive propositions, when the symbols are given appropriate interpretations, nevertheless give facts in logic, and the theorems are, indeed, as can be verified, logical consequences of the primitive propositions. It seems, therefore, that it should be possible, by merely changing the logical structure of the theory, to transform it into a mathematical, an abstract science. This indeed, is the case. And this transformation I wish now to carry out. The transformation will bring out very clearly the true mathematics underlying the theory.

We have seen that the ideas denoted by p, $\sim p$, $p \vee q$ and the propositions *1·7 and *1·71 are quite suitable as primitives for a mathematical theory, but that the notion ϕx and the proposition *1·72 are to be rejected as primitives on the ground that ϕx is definable by means of p, $\sim$, $\vee$, and that *1·72 is derivable from *1·7 and *1·71. In order to change the theory into a mathematical science, therefore, we must so dispose of the remaining primitives, the idea $\vdash \cdot p$ and the propositions *1·1 — *1·6, as to free them from the objections raised against them, without changing the facts of logic embodied in the present symbolism. The necessary changes are simple.

The symbol $\vdash \cdot p$ stands for the proposition "p is true." This proposition implies that there exists a certain proposition, "1" say, characterized as "true," to which p is equivalent. Let "$p=q$" denote the proposition "p and q may be interchanged." Then, if p and q are propositions of our theory, "$p=q$" is the proposition "p and q are equivalent," that is, "p and q are both true or both false." And so the proposition "p is true" may be expressed by the symbolism "$p=1$," wherein, it is to be noted, p and 1 are ideas of our theory, but "$=$" is not. The new symbolization of "p is true" thus removes $\vdash \cdot p$ from the list of primitives. The new symbolization of "p is true" at the same time properly disposes of the propositions $*1\cdot 2-*1\cdot 6$. For each of these propositions is changed to the form $\phi(p, q, \cdots)=1$, in which $\phi(p, q, \cdots)$ and 1 are ideas of the theory and "$=$" is not, and so is free from the criticism expressed in the preceding section.

As to proposition $*1\cdot 1$, though the authors say that they cannot express it symbolically, they *use* the proposition as if it were written "From '$\vdash \cdot p$' and '$\vdash \cdot p \supset q$' follows '$\vdash \cdot q$'," that is, "From $p=1$ and $\sim p \vee q=1$ follows $q=1$." And so $*1\cdot 1$, too, can be stated in a form free from the above criticism.

I have now indicated how to translate the theory into the language of an abstract mathematical science. Let me actually perform the translation. In this translation, I shall, for the sake of easy comparison of the theory with Boolean logic, replace $p \supset q$ by $\sim p \vee q$, in accordance with $*1\cdot 01$, and shall write p' for $\sim p$ and $p+q$ for $p \vee q$. For the sake of easy reference, the original numbering of the propositions will be retained, except that the sign "*" before the numbers will be omitted. The theory restated as a mathematical science follows.

Consider an undefined *class* K of elements $p, q, r, \cdots$. Let p' be the result of an undefined *unary operation* on a K-element p, and $p+q$ the result of an undefined *binary operation* on the K-elements p, q. The theory of deduction is the *system* $(K, ', +)$ satisfying the postulates $1\cdot 1-1\cdot 71$ below. In postulates $1\cdot 1-1\cdot 6$ there is implied the supposition that the indicated elements in each proposition are K-elements. In each of the propositions $1\cdot 2-1\cdot 6$ there is also implied the supposition that the element 1 of postulate $1\cdot 1$ exists and is unique. The postulates follow.

1·1. There exists a K-element 1 such that from $p=1$ and $p'+q=1$ follows $q=1$.

1·2. $(p+p)'+p=1$.

1·3. $q'+(p+q)=1$.

1·4. $(p+q)'+(q+p)=1$.

1·5. $[p+(q+r)]'+[q+(p+r)]=1$.

1·6. $(q'+r)'+[(p+q)'+(p+r)]=1$.

1·7. If p is a K-element, p' is a K-element.

1·71. If p and q are K-elements, $p+q$ is a K-element.

To complete the statement of the theory in the new language, I add the authors' definitions of $p \supset q$, $p \cdot q$ (or pq), and $p \equiv q$:

Def. 1·01. $(p \supset q)=p'+q$.

Def. 3·01. $pq=(p'+q')'$.

Def. 4·01. $(p \equiv q)=(p \supset q)\ (q \supset p)=(p'+q)\ (q'+p)$.

We have now the "mathematicized" form of the theory of deduction. In this form the propositions give the same facts of logic as the propositions of the old form, but there is no confusion in it between the ideas belonging to the theory and the ideas that are outside it, and the ideas and propositions in it are all of the type of the ideas and propositions found in any orthodox mathematical theory.

6. *Conclusion.* The foregoing discussions make clear the mathematical nature of Whitehead and Russell's theory of deduction. The theory as a mathematical science is the system $(K, ', +)$ satisfying the postulates 1·1 − 1·71 of the preceding section. This system is the theory of the *Principia* cleared of obvious redundancies in the primitives and written in the notation of Boolean logic. The two forms of the theory give the same facts of logic; but while in the Boolean form the ideas that are outside the theory are separated from those that belong to the theory, in the *Principia* form this separation is not made. It is the failure to distinguish between these two sets of ideas that makes the authors say that "the recognized methods of proving independence is not applicable, without reserve, to fundamentals." The theory, as seen from the Boolean form, is not more fundamental than any other mathematical theory and is subject to postulational investigations applicable to all other mathematical sciences.

THE UNIVERSITY OF CALIFORNIA

OFFICIAL COMMUNICATIONS

Meetings of the Society have been fixed at the following times and places:

MINNEAPOLIS, MINNESOTA, SUMMER MEETING AND COLLOQUIUM, September 8-11, 1931.

Abstracts must be in the hands of Associate Secretary M. H. Ingraham, University of Wisconsin, Madison, Wis., not later than August 10. For publication in advance of the meeting, abstracts must be received not later than July 7. Professor Marston Morse will deliver a series of colloquium lectures on *The calculus of variations*. Professor C. C. MacDuffee will deliver an address entitled *Ideals in the theory of linear algebras*.

NEW YORK CITY, October 31, 1931.

Abstracts must be in the hands of Associate Secretary Tomlinson Fort, 501 West 116th Street, New York City, not later than October 10. In order to be printed in the Bulletin in advance of the meeting, abstracts must be in the hands of the Associate Secretary not later than September 7. By invitation of the program committee, Professor D. V. Widder will deliver an address on *Some recent developments in the theory of Laplace integrals*.

COLUMBUS, OHIO, November 27-28, 1931.
Abstracts are due by November 5, 1931.

CALIFORNIA INSTITUTE OF TECHNOLOGY, PASADENA, November 28, 1931.
Abstracts are due by October 31, 1931.

NEW ORLEANS, LOUISIANA, ANNUAL MEETING, December 28-31, 1931.
Abstracts are due by November 28, 1931.

(The usual February Meeting in New York City will be omitted.)

NEW YORK CITY, March 25-26, 1932.
Abstracts are due by March 5, 1932.

STANFORD UNIVERSITY, CALIFORNIA, April 9, 1932.
Abstracts are due by March 12, 1932.

R. G. D. RICHARDSON, *Secretary of the Society*.

Articles for insertion in the BULLETIN should be addressed to E. R. HEDRICK, Editor of the BULLETIN, University of California at Los Angeles. Reviews should be sent to W. R. LONGLEY, Yale University, New Haven, Conn. Notes should be sent to H. W. KUHN, Ohio State University, Columbus, Ohio.

Subscriptions to the BULLETIN, orders for back numbers, and inquiries in regard to non-delivery of current numbers should be addressed to the American Mathematical Society, 450-459 Ahnaip St., Menasha, Wis., or 501 West 116th St., New York.

The initiation fees and the annual dues of members of the Society (see this BULLETIN, p. 322, May, 1930; and the List of Officers and Members, October, 1930, p. 58), are payable to the Treasurer of the Society, G. W. Mullins, 501 West 116th St., New York.

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For official Communications and Notices, see the inside of the back cover.

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